FIRST COURSE IN ALGEBRA

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PREFACE

THE present book forms Volume I of a two-volume series on high school algebra and embodies an especial effort to connect the elements of algebra in a clear and forcible manner with the affairs of every-day life. To this end a large variety of applied problems have been introduced and an unusual number of illustrative diagrams used in connection with both the reading matter and the exercises. At the same time, the text follows a distinctly logical line of development, thus affording that drill in accurate thought and expression which should characterize all the mathematical work of the high school. In brief, the presentation is intended to bring algebra into most intimate connection with nature while preserving its fundamental disciplinary values. This, we believe, is the demand, and properly so, of teachers and educators to-day.

Among the special features may be noted the abundance of exercises and the care with which they have been selected and graded. Problems of more than average difficulty have usually been accompanied with a hint in order that the pupil's time and energy may be reasonably conserved. At the end of the volume will be found an extensive list of supplementary exercises suitable for class use, in addition to those given in the body of the text. It may be noted also that, as in the authors' *Plane and Solid Geometry,* the book is so arranged that certain topics may be omitted; at the discretion of the teacher, without interfering with the

continuity of the whole. Such topics have been accompanied by the star (*).

Considering the various chapters in some detail, Chapter I acquaints the pupil with certain elementary notions that are central in algebra, more especially that of the literal number and the simple equation. The development at this point is easy and natural, being based upon familiar simple principles from arithmetic, and the whole is abundantly supplied with problems of a kind that will instinctively appeal to the average beginner, and thus invite his early interest in the subject. Chapter II, where negative number first appears, contains an unusually large variety of illustrations tending to bring out the full significance of such numbers. Chapter IV, entitled *Multiplication and Special Cases of Factoring*, departs slightly from the usual procedure in that it develops the elements of factoring along with those of multiplication instead of delaying the entire subject of factoring for a later chapter. In this way the two subjects (which are but the reverse of each other) are seen from the beginning in their mutual relations. However, it is only the simpler types of factoring that are taken up at this early stage, the more difficult ones being reserved for the following chapter. The result of this arrangement, as shown by experience, is that factoring, having thus had an early and natural beginning in connection with multiplication, becomes less isolated and consequently is the more readily grasped by the beginner.

Passing to the later chapters, it may be observed that in the treatment of surds, radicals, and roots (Chapters XIV and XV) the pupil is taught the use of tables — a topic heretofore quite neglected in elementary algebra, yet one of increasing importance owing to the number of pupils that pass from our high schools directly into technical pursuits.

A word should perhaps be said here regarding the treat-

ment of graphical methods that the book presents. The authors believe that while such methods may be (and indeed often are) introduced into algebra at a very early stage, yet the most reliable experience indicates that the pupil has as much as he can reasonably be expected to do at the very outset if he masters merely the meaning and technique of the literal number and simple equation. This much he can do immediately from his knowledge of arithmetic, while graphical interpretation introduced at this stage is apt to seem artificial. Graphics are therefore postponed until Chapter XII, at which time the pupil has a substantial groundwork in algebraic facts and reasoning.

Volume II begins with a systematic review of the fundamental processes of algebra. This is followed by a more extended treatment of certain of the topics in Volume I and this in turn by chapters upon a number of the more advanced topics required for entrance into our best colleges and technical schools.

In conclusion the authors would here acknowledge their indebtedness to various other texts, especially the recent treatises of Godfrey and Siddons appearing in England, and to various current discussions and articles such as have appeared from time to time in SCHOOL SCIENCE AND MATHEMATICS and in the AMERICAN MATHEMATICAL MONTHLY. The authors are also indebted to Professor L. C. Karpinski for the sketch of the history of algebra appearing at the opening of the book, and they would here express their gratitude to certain friends who have kindly examined the manuscript and proof sheets and offered timely suggestions and criticisms — in particular to Professor E. R. Hedrick.

WALTER B. FORD.
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FOREWORD TO THE PUPIL

ALGEBRA is not a difficult subject, but you must remember to read carefully what the book says. Every sentence is important, and if you fail to understand one sentence you will be quite likely to have difficulty from that point on through the later pages. There are times perhaps when, after repeated efforts, you cannot understand at a certain point — at such times you should freely express your difficulty to the teacher, who will explain the meaning and help you.

Remember also that it is best to examine carefully the *solution of the problems* worked out in the book before you attempt to do other problems like them. For example, before attempting the problems on page 138 you should have examined carefully those of a similar kind that are completely worked out on page 137.

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HISTORICAL INTRODUCTION

BY LOUIS C. KARPINSKI

ALGEBRA had its beginnings in a very remote period of history, probably not less than 3500 years ago. We know, in fact, that at about that time a certain Egyptian named *Ahmes* (pronounced Ā′-mes) wrote a mathematical text-book in which he proposed several problems containing equations.* For example, one of the problems reads as follows: "An unknown and its seventh make 19. What is it?" In solving this, Ahmes did not use a letter to represent the unknown, as we would naturally do now, but the steps he followed were nevertheless essentially algebraic. Other Egyptian texts containing similar problems are preserved to-day in the museums of Paris and Berlin.

During the era of Greek civilization and culture, which followed that of ancient Egypt, algebra made but little progress because the Greeks were interested chiefly in geometry. However, *Euclid* (Ū′-klid), who lived about 300 B.C., and *Archimedes* (Ark-i-mē′dēs), who lived at Syracuse on the island of Sicily about 250 B.C., both of whom were great mathematicians of this period, could solve first-degree equations and even quadratics, but they used geometric methods instead of the simpler algebraic methods which we now have.

* Ahmes' work is still preserved and is to-day in the British Museum. Like all manuscripts of such antiquity, it is written upon papyrus, or paper made by pasting together long thin strips cut from the papyrus plant.

Following the Greeks, the Romans had little or no influence upon the development of mathematics. In fact, we find little progress in algebra except in India until about 800 A.D., at which time the leading scientists of the world were among the Arabs. In particular, the Arab mathematicians familiarized themselves with the writings of the Greeks and Hindus on algebra, and made advances over what they had thus received. As a result, the first systematic treatise on algebra appeared, being written by an Arab named *Mohammed ibn Musa al Khowarizmi*. Here it is shown how algebra may be applied to the solution of certain geometric problems and to certain engineering questions. It may be remarked that the same author also wrote a famous treatise on arithmetic, which was long used. This work brought into Europe for the first time the numeral system (called the Hindu-Arabic system) which we use to-day and which employs the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 instead of earlier symbols, such as the *I*, *V*, *X*, *C*, and *M* used by the Romans.

By the year 1200 A.D. many of the Arabic works dealing with science had been translated into Latin and it was chiefly through such translations that algebra became known to the Europeans of the Middle Ages.

Modern algebra may be said to have begun with the French statesman *Viète* (Vē-ĕt′, 1540–1603), who was the first to use letters to represent known and unknown quantities. He used vowels to represent unknowns and consonants to represent knowns. Our common use of x, y, and z to represent unknowns arose with the great French mathematician and philosopher *Descartes* (Dā-cart′, 1596–1650) and was adopted soon afterward by the great English mathematician and astronomer *Sir Isaac Newton* (1642–1727) and by the great German philosopher and mathematician *Leibnitz*

(Līb′-nitz, 1646–1716). Portraits of these three men with brief notes about them occur in this book.

As to the symbols +, −, =, etc., these came into use only after the invention of printing, being introduced as convenient abbreviations. For example, the signs + and − first occur in a German text published in 1489, while the sign = was first used by the English physician *Robert Recorde* (1510–1558), who published the first English work on algebra, calling it the "Whetstone of Witte" (whetstone of wit).

Detailed accounts of the origin and development of algebra may be found in histories of mathematics, *e.g.* BALL, *A Short History of Mathematics;* CAJORI, *History of Elementary Mathematics;* etc.

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NEWTON

(*Sir Isaac Newton*, 1642–1727)

Discoverer of the law of gravitation and famous in algebra for his discovery of the binomial theorem. Inventor of the branch of higher mathematics called the Calculus, wherein rates of motion and other changing, or variable, quantities are extensively studied.

FIRST COURSE IN ALGEBRA

CHAPTER I

LITERAL NUMBERS

1. The Use of Letters. In arithmetic we often find it convenient to let a letter stand for a number. Thus, if we let I stand for the interest on \$125 for 2 years at 5 %, we may write

$$I = \$125 \times 2 \times .05 = \$12.50. \quad \textit{Ans.}$$

In Algebra the use of letters in this way is very common. Let us take another example. We know that the area of a rectangle may always be found by multiplying its length by its breadth. To express this rule by the use of letters we have only to let A stand for *area*, l for *length*, and b for *breadth*, and then the rule becomes simply

$$A = l \times b.$$

Observe that this gives the right value for A in *every* case. Thus, if we suppose $l = 2$ ft. and $b = 3$ ft., it gives instantly

$$A = 2 \times 3, \text{ or } 6 \text{ sq. ft.,}$$

which we know by arithmetic is the right answer.

The advantage in using letters is twofold. It enables us to write general rules and it is of great assistance in solving problems, as we shall soon see.

2. Literal Numbers. Numbers that are represented by letters are called ***literal numbers.***

ORAL EXERCISES

1. State by use of letters the following rule: The volume of a box is equal to the product of its length, its width, and its height.

[HINT. Let V stand for volume, l for length, w for width, and h for height.]

2. Show that your result for Ex. 1 gives the correct answer when the length is 4 ft., the width 3 ft., and the height 2 ft.

3. Symbols. The symbols $+$, $-$, $\times$, and $\div$ are used with literal numbers as with other numbers, and their meanings are as in arithmetic.

Just as $8+5$ means the sum of 8 and 5, so $a+b$ (read *a plus b*) means the sum of a and b. Likewise, just as $8-5$ means the result of subtracting 5 from 8, so $a-b$ (read *a minus b*) means the result of subtracting b from a. Again, just as 8×5 means the product of 8 and 5, so $a\times b$ (read *a times b*) means the product of a and b. Finally, $a\div b$ (read *a divided by b*) means in all cases the quotient of a divided by b.

Besides the rules just stated, there are others in algebra which must now be carefully noted:

Instead of $2\times a$ it is customary to write simply $2a$. In the same way, $3\times a$ is written $3a$, and $4\times x$ is written $4x$, etc. In fact, if a and b stand for any numbers, then $a\times b$ is written in the simple form ab.

But notice that for 2×3 we cannot write 23, since this means twenty-three. What we have said applies only to *literal* numbers.

Sometimes $a \times b$ is written in another form, namely $a \cdot b$. Observe that the dot is here placed just *above* the line.

If a stands for 2, then it follows from what we have just said that $5a$ and $5 \cdot a$ mean the same thing, namely 5×2, or 10. In the same way, if x stands for 2 and y stands for 6, then xy and $x \cdot y$ both mean 2×6, or 12.

Again, we know from arithmetic that when any two numbers are multiplied together it makes no difference which is taken as the multiplier and which as the multiplicand. Thus, $2 \times 3 = 3 \times 2$, and $7 \times 9 = 9 \times 7$, etc. Since a is a number, it follows that $2 \times a = a \times 2$, and each of these expressions is written $2a$.

The quotient $a \div b$ is frequently written in the form $\frac{a}{b}$, or a/b.

ORAL EXERCISES

1. If a tennis ball costs 35 cents, what do 5 such balls cost?

2. If a tennis ball costs c cents, what do 5 such balls cost? *Ans.* $5c$ cents.

3. If a tennis ball cost r cents last year and the price has advanced 10 cents, what is the present cost?

Ans. $r + 10$ cents.

4. Since 3 feet make one yard, how many feet are there in 5 yards? in 65 yards? in n yards?

5. Since there are 16 ounces in one pound, how many ounces in 7 pounds? in n pounds? in r pounds?

6. How many minutes in 4 hours? in 30 hours? in b hours?

7. Give the expression that represents the number one greater than d. *Ans.* $d + 1$.

8. Give the expression that represents the number 10 less than c.

9. State all the ways in which the following can be written without the use of the sign $\times$.

(a) $7 \times x$ (c) $y \times 9$ (e) $m \times 9$ (g) $P \times Q$
(b) $x \times 7$ (d) $9 \times y$ (f) $m \times n$ (h) $10 \times F$

10. State the value of each of the following when $a=2$.

(a) $1+a$ (b) $3-a$ (c) $6a$ (d) $\frac{a}{2}$ (e) $3a-2$

11. If it takes x minutes to walk to school and 10 minutes to return, what is the total time occupied?

12. A boy plans to go hunting. If it takes him x minutes to go to the station and 5 times as long on the train, how long will it take him to go where he expects to hunt?

13. A man has d dollars invested in a farm and three times that amount in bonds. How much has he invested in all?

14. If, in Ex. 13, $d=\$50{,}000$, how much has he invested in bonds?

15. Three bundles of shingles are required for every 100 square feet in a roof. How many bundles of shingles are required for a roof containing 900 square feet? How many bundles are required for a roof of n square feet?

16. The cost of a baseball bat is 6 times that of a ball. If the bat costs b cents, what is the price of the ball?

17. If a man rides a certain distance in 10 hours, what part of the distance does he ride in 1 hour? in 7 hours? in h hours?

18. If A can do a piece of work in 3 hours, what part of it can he do in 1 hour? in 2 hours? in r hours? in $r+5$ hours?

19. If A's age is $3r$ and B's age is 4 times A's, what represents B's age?

20. The top of my desk contains $6b$ square inches. If yours is one-fourth as large, what represents the number of square inches in yours?

21. In two quarters there are 2×25 cents. How many cents in d quarters? in r quarters?

22. If a train runs m miles in 1 hour, how many miles will it run in 3 hours? in y hours? in $3y$ hours?

23. What is the next whole number after 12? If x is a whole number, what represents the next whole number after x?

24. If y is an *even* number, what is the next even number after y?

25. If z is an *odd* number, what is the next odd number after z?

26. If z is an odd number, what is the next even number after z?

27. Find the value of each of the following when $a=1$ and $b=2$.

(*a*) $a+b$	(*c*) $b-a$	(*e*) $2b-a$
(*b*) $2a+b$	(*d*) ab	(*f*) $3ab$

For further exercises on this topic, see the review list, p. 20, *and Appendix, p.* 289.

4. Factor. In arithmetic when two or more numbers are multiplied together each is called a ***factor*** of the product. Thus, in the product $5 \times 3 \times 2$ the factors are 5, 3, and 2. The same is true in algebra. Thus, the factors of $3a$ are 3 and a; the factors of $2xy$ are 2, x, and y.

5. Coefficient. In algebra whenever a number is separated into *two* factors either is called the ***coefficient*** of the other. Thus, in the expression $3a$, 3 is the coefficient of a, or it may be said also that a is here the coefficient of 3. Likewise, in $15ab$ the coefficient of ab is 15.

ORAL EXERCISES

Name the factors in each of the following expressions.

1. $5x$. **2.** $7b$. **3.** $11bx$. **4.** $19xyz$. **5.** $29mn$. **6.** $31pqrs$.

State the coefficient of x in each of the following expressions.

7. $5x$. **8.** $13x$. **9.** $19ax$. **10.** $7abx$. **11.** $19pqxr$.

12. Explain the difference between 2.3 and $2 \cdot 3$; between 2.13 and $2 \cdot 13$.

13. State the simplest form for each of the following expressions.

(*a*) 3×16.	(*e*) $10 \times 3a$.	(*i*) $10 \times 5z$.	(*m*) $\frac{1}{2}$ of $12a$.
(*b*) $2 \times 3 \times 5$.	(*f*) $3a \times 10$.	(*j*) $10 \times k$.	(*n*) $\frac{1}{6}$ of $24x$.
(*c*) $8 \times a$.	(*g*) $6 \times 4r$.	(*k*) $\frac{1}{2} \times k \times 10$.	(*o*) $\frac{2}{3}$ of $6rs$.
(*d*) $8 \times 4a$.	(*h*) $4r \times 6$.	(*l*) $6 \times \frac{1}{3} \times k$.	(*p*) $\frac{3}{5}$ of $10abc$.

14. State the simplest form for each of the following expressions.

(*a*) $12ar \cdot 5$	(*d*) $24b \cdot 7c$	(*g*) $6 \cdot 5rs \cdot 3$	(*j*) $2 \cdot xyz$
(*b*) $16abc \cdot 2$	(*e*) $9rs \cdot 6$	(*h*) $\frac{1}{2}bc \cdot 6$	(*k*) $6mn \cdot 3p$
(*c*) $5axy \cdot q$	(*f*) $14a \cdot y$	(*i*) $\frac{1}{4}a \cdot b \cdot 20$	(*l*) $5 \cdot 10yz$

15. Sometimes the product of two factors is given together with one of the factors, and it is required to find the other factor. This leads, as in arithmetic, to *division*. Thus, if the given product is 27 and the given factor is 9, the other factor is $27 \div 9$, or 3.

In each of the following expressions, state what the given product is, what the given factor is, and what the other factor would be.

(*a*) $81 \div 9$.	(*c*) $21x \div 3$.	(*e*) $6m \div 3$.	(*g*) $10ab \div 5$.
(*b*) $16 \div 4$.	(*d*) $18c \div 18$.	(*f*) $75q \div 25$.	(*h*) $125xyz \div 25$.

6. The Equation. We are now ready to see how the literal number may be used in solving problems. Here the advantage of using letters, mentioned in § 1, will become clearer.

Example 1. If twice a certain number is increased by 5, the result is 29. What is the number?

Solution. Let x stand for the number sought. Then, from what the problem says, we must have

$$2x+5=29.$$

This statement is called an ***equation***, since it is an equality between two numbers. It may be compared to a balance (see Fig. 1), on one side of which is $2x+5$ and on the other side is 29.

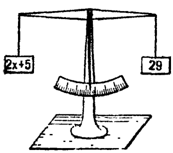


Fig. 1.

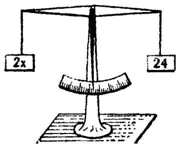


Fig. 2.

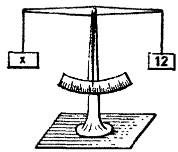


Fig. 3.

Now, the balance will, of course, remain undisturbed if we subtract 5 from *each* side. The result (see Fig. 2) when stated in the form of a new equation is

$$2x=24.$$

Let us now divide both sides by 2. This gives (see Fig. 3) the new equation

$$x=12.$$

Therefore, the number sought is 12. *Ans.*

Condensed Solution. The work which we have just done in finding x may be greatly condensed. All we really need to do is to write down three steps as follows:

From the problem, we know, as before, that $2x+5=29$.

Subtracting 5 from both sides, we find $2x=24$.

Dividing both sides of last equation by 2, gives $x=12$. *Ans.*

We shall now solve another example, but we shall condense the steps in this way from the beginning.

EXAMPLE 2. A person wishes to find the weight of a tennis ball. He puts three such balls in the left scale pan of the balance and six ounces in the right pan. He finds this too much, but by adding 1 ounce to the left pan he secures a good balance. How much does one ball weigh?

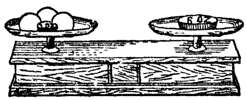


FIG. 4.

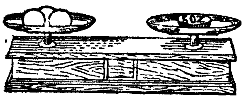


FIG. 5.

SOLUTION. Let x stand for the number of ounces in one ball. Then, from what the problem says, we must have

$$3x+1=6.$$

Subtracting 1 from both sides, we find

$$3x=5.$$

Dividing both sides by 3, we have

$x=1\frac{2}{3}$, *the number of ounces each ball weighs.* *Ans.*

CHECK. Whenever an answer does what the problem demands of it, it is said to *check*. In this problem, the answer $1\frac{2}{3}$ checks because $3\times1\frac{2}{3}+1=6$, just as the first equation (or $3x+1=6$) demands.

WRITTEN EXERCISES

Solve the following exercises, letting x stand for the unknown number. Condense the work as in § 6 and check each answer.†

1. If three times a certain number is increased by 2 the result is 20. What is the number? *Ans.* 6.

† It is true that these exercises, like those in § 6, may be worked by arithmetic, but you should work them by algebra in order to be prepared for the more difficult problems that are coming later, which cannot be worked easily by arithmetic.

2. A boy paid $1.25 for a base ball bat and three balls. The price of the bat was 50 cents. What was the price of a ball?

[HINT. Work in cents.]

3. A dozen eggs are placed in a basket that weighs 6 oz. and the whole is then found to weigh 30 oz. What is the weight of one egg?

4. A wagon loaded with 500 paving bricks weighs 1250 lb., and the empty wagon weighs 500 lb. Find the weight of a single brick.

5. A jar containing 40 cakes of chocolate weighs 44 lb. The jar alone weighs 6 lb. Find the weight of one of the cakes of chocolate.

6. A wall twelve feet high has 64 layers of mortar and 64 courses of brick. If each brick is 2 in. thick, find the thickness of each layer of mortar.

7. A box containing a ream (20 quires) of paper weighs 3 lb. The box alone weighs 8 oz. Find the weight of one quire of paper.

Find the weight of a single sheet of this paper, if 1 quire = 24 sheets.

8. A stick ten feet long is cut into two pieces, one of which is 2 feet longer than the other. Find the length of each piece.

9. Two hunters together shot thirty quail, but one of them shot four more than the other. How many quail did each shoot?

10. Three basket ball teams played fifteen games. One team won three games and the other two teams divided the remaining victories equally between them. How many games did each of the other teams win?

7. Adding to Both Sides of an Equation. If we wish to solve such an equation as

$$3x - 1 = 14,$$

it is necessary to *add* 1 to both sides. This can be done since the balance is not thereby disturbed. Adding 1 to both sides, we find

$$3x = 15.$$

Dividing both sides by 3, we have

$$x = 3. \quad \textit{Ans.}$$

ORAL EXERCISES

In order to solve certain of the following equations, it is necessary to *subtract* the same number from both sides; in other equations it is necessary to *add* the same number to both sides. State which should be done in each case.

1. $x - 3 = 10.$
2. $x + 4 = 7.$
3. $x - 8 = 16.$
4. $x + 10 = 17.$
5. $r + 32 = 50.$
6. $q - 5 = 3.$
7. $2s + 19 = 81.$
8. $2x - 7 = 13.$
9. $2k + 101 = 201.$
10. $5c + 3\frac{3}{4} = 6\frac{1}{4}.$
11. $12k - 12 = 12.$
12. $x + 7\frac{1}{2} = 12\frac{3}{4}.$
13. $x - 6\frac{2}{3} = 25.$
14. $.3x + .4 = .9$
15. $.1x + .65 = .75$
16. $4p - 13 = 27.$
17. $7x - 21 = 42.$
18. $\frac{1}{3}m - 2 = 8.$

WRITTEN EXERCISES

Solve each of the following equations; that is, find the value of the unknown letter. Check each answer.

1. $2x - 1 = 5.$
2. $2x + 1 = 5.$
3. $4x - 3 = 17.$
4. $5x + 3 = 18.$
5. $2x - 1\frac{1}{2} = \frac{1}{2}.$
6. $1.2x + 2.3 = 4.7$
7. $5x - 1.5 = .75$
8. $5x + .03 = .08$
9. $11s - \frac{3}{4} = \frac{5}{8}.$
10. $t - 1\frac{5}{6} = 2\frac{2}{3}.$
11. $8r + .23 = .87$
12. $z - 8\frac{4}{5} = 10\frac{1}{5}.$
13. $3g + .6 = \frac{3}{4}.$
14. $1\frac{1}{2}d + 7 = 22.$
15. $3.7x - 7.4 = 0.$

16. If I subtract 18 from a certain number, the result is 4. Write an equation expressing this fact and solve it to find the number.

17. If 16 is subtracted from three times a certain number the result is 110. Find the number.

18. Our team played ball with the seventh grade. Our score was 3 less than twice theirs. Our score was 11. What was theirs?

19. In paying for $3\frac{1}{4}$ yards of cloth a lady gave the clerk a \$2 bill and received 57 cents back in change. How much was the material a yard?

20. A man said, "For \$2500 you can have a half interest in my business, and I will give you my automobile, which I value at \$500, in the bargain." At what figure did he value his business?

For further exercises on this topic, see the review list, p. 20, *and Appendix, p.* 290.

8. Multiplying both Sides of an Equation. We have already seen how we may divide both sides of an equation by the same number. Thus, if

$$2x = 10,$$

we simply divide both sides by 2 to get x, obtaining

$$x = 5. \quad \textit{Ans.}$$

Likewise, we may at any time *multiply* both sides of an equation by the same number, since the balance is not thereby disturbed. For example, to solve the equation,

$$\frac{x}{3} = 5,$$

we multiply both sides by 3, thus obtaining

$$x = 15. \quad \textit{Ans.}$$

ORAL EXERCISES

Find the value of the letter in each of the following equations.

1. $\frac{x}{3}=10.$
2. $\frac{r}{5}=16.$
3. $\frac{b}{8}=12.$
4. $\frac{p}{2.5}=10.$
5. $\frac{a}{21}=20.$
6. $\frac{A}{15}=60.$
7. $\frac{S}{13}=12.$
8. $\frac{x}{1.4}=5.$
9. $\frac{x}{22}=7.$
10. $\frac{r}{4}=12.$
11. $\frac{y}{3}=15.$
12. $\frac{z}{5}=22.$

WRITTEN EXERCISES

Find the value of the letter in each of the following equations.

1. $\frac{x}{4}=10.$
2. $\frac{y}{17}=20.$
3. $\frac{z}{15}=10\frac{1}{3}.$
4. $\frac{x}{3.1}=62.$
5. $\frac{y}{8.9}=2670.$
6. $\frac{z}{3.25}=\frac{2}{3}.$
7. $\frac{p}{8.1}=\frac{7}{9}.$
8. $\frac{x}{3.5}=1.7$
9. $\frac{r}{.12}=.08$

10. Find the value of x in the equation $\frac{2x}{3}=4.$

SOLUTION. Multiplying both sides by 3,

$$2x=12.$$

Dividing both sides by 2,

$$x=6. \quad Ans.$$

Find the value of x in each of the following equations.

11. $\frac{3x}{4}=9.$
12. $\frac{5x}{7}=3.$
13. $\frac{2x}{1.1}=4.$
14. $\frac{3x}{.08}=12.$
15. $\frac{2x}{5}=8.$
16. $\frac{3x}{7}=21.$

17. A certain number divided by 13 gives 50. Write an equation expressing this fact and solve it to find the number.

18. What number divided by .4 gives 4? (Work by algebra.)

19. If twice a certain number is divided by 3, the result is 12. What is the number? (Compare with Ex. 10.)

20. By what number must $7\frac{1}{2}$ be multiplied to give 425?

21. The diameter of an ear of corn is about one fourth of its length. What is the length if the diameter is $1\frac{3}{4}$ inches?

9. Axioms. Our study of the equation has now brought out four different principles, which are called ***axioms***. As we shall need these often, they should be committed to memory.

AXIOM I. *Equal amounts added to equal amounts give equal amounts.*

AXIOM II. *Equal amounts subtracted from equal amounts give equal amounts.*

AXIOM III. *Equal amounts multiplied by equal amounts give equal amounts.*

AXIOM IV. *Equal amounts divided by equal amounts give equal amounts.*

10. Common factor. In arithmetic a number which is a factor of two or more numbers is called a ***common factor*** of those numbers. Thus, 5 is a common factor of 10, 15, and 20.

The same is true in algebra. Thus, x is a common factor of $3x$, $8x$, and $11x$.

11. Like Numbers. Literal numbers that have a common factor are called ***like numbers***. Thus, $3x$, $8x$, and $11x$ are like numbers, as also $2a$, $7a$, and $9a$.

Like numbers are very easily added together. Thus, we evidently have $2x+3x=5x$; that is, all we need to do is to add the coefficients to obtain a new coefficient, then multiply the result by the common factor.

Similarly, we have $5x-3x=2x$. Thus, we see that in order to subtract like numbers all we need to do is to subtract the coefficients to obtain a new coefficient, and then multiply the result by the common factor.

ORAL EXERCISES

State the simplest form for each of the following expressions.

1. $2a+4a$.
2. $4a-3a$.
3. $3x+3x+4x$.
4. $10y+2y+y$.
5. $10y+2y-y$.
6. $10y-2y-y$.
7. $16x-8x+3x-x$.
8. $25x+4x-3x-2x$.
9. $6m-2m-3m-m$.
10. $\frac{3}{4}x-\frac{1}{2}x$.
11. $5.2x+4.3x$.
12. $5.2y+4.3y$.

12. Further Study of the Equation. Sometimes we have to solve such an equation as

$$5x=16-3x.$$

This is different from any we have considered before because the x appears on *both* sides. If we remember the axioms of § 9, however, we can easily solve it. The steps are as follows:

$$5x=16-3x.$$

Adding $3x$ to both sides,

$$8x=16. \qquad \text{(Axiom I)}$$

Dividing both sides by 8,

$$x=2. \quad \textit{Ans.} \qquad \text{(Axiom IV)}$$

Again, suppose we wish to solve the equation

$$6+6x=3x+12.$$

Here we proceed as follows. Subtracting $3x$ from both sides,

$$6+3x=12. \qquad \text{(Axiom II)}$$

Subtracting 6 from both sides,

$$3x=6. \qquad \text{(Axiom II)}$$

Dividing both sides by 3,

$$x=2. \quad \textit{Ans.} \qquad \text{(Axiom IV)}$$

Thus, the axioms in § 9 enable us to solve all such equations. We may sum up our results in the following rule.

RULE FOR SOLVING EQUATIONS. *Use axioms I and II to get rid of all the terms, or parts, on one side of the equation that contain the letter, and to get rid of all terms, or parts, on the other side of the equation that do not contain the letter. Then use axioms III and IV to find the value of the letter.*

ORAL EXERCISES

In each of the following equations, tell what to do to get rid of the term on the right side that contains the letter.

1. $3=5-x.$

2. $4=20-2x.$

3. $5x=3x+10.$

4. $6r-2=2r+10.$

5. $10+5c=30-15c.$

6. $5m+2=6+m.$

7. $3y-1=y+2.$

8. $8z-\frac{1}{2}=3-z.$

9. $\frac{2}{3}s+4=\frac{1}{3}s+8.$

10. $2\frac{1}{2}a-5=\frac{1}{2}a-2.$

11. $.5x=.2x+.3$

12. $.5x=\frac{1}{3}x-.08$

In each of the following equations, tell what must be done to the equation to get rid of the term on the left side that contains no letter.

13. $4x+3=15.$

14. $3r-8=16.$

15. $5+4a=21.$

16. $10+6z=3z+40.$

17. $4x-1=2x+6.$

18. $\frac{1}{2}r+\frac{1}{3}=\frac{1}{3}r+\frac{2}{3}.$

19. $5m-\frac{1}{4}=2m+\frac{3}{4}.$

20. $\frac{3}{4}r-8=\frac{2}{3}r+3.$

21. $6m-2\frac{3}{4}=m+2.$

WRITTEN EXERCISES

Solve each of the following equations and check your answer in each case.

1. $5x-2=x+10$.
2. $10r+4=6r+16$.
3. $4+5a=11-2a$.
4. $13y-99=7y+69$.
5. $6z-4=2z+20$.
6. $12x-2x=20+5x$.
7. $20+7x-5x=48-4x$.
8. $6x-15-9=4x-2x$.
9. $2x+3x+5-x=8+x$.
10. $\frac{1}{2}x+2x-3=x+1$.
11. $\frac{4}{3}x+1=\frac{1}{3}x+2$.
12. $.3x+5=4x+1.3$

13. Find the number which added to three times itself gives 36.

SOLUTION. Let x = the number sought.

Then,
$$3x+x=36,$$
hence,
$$4x=36,$$
and therefore,
$$x=9. \quad \textit{Ans.}$$

14. If 8 is subtracted from 5 times a certain number, the result is equal to the number increased by 2. What is the number? (See Ex. 13.)

15. The water and steam in a boiler occupied 120 cu. ft., and the water occupied twice as much space as the steam. How many cubic feet did each occupy?

[HINT. After finding x, the space occupied by the steam, it is necessary to multiply it by 2 to find the space occupied by the water.]

16. In a fire, B lost twice as much as A, while C lost three times as much as A. If their combined loss was \$6000, what was the loss of each?

17. A, B, and C begin business with a capital of \$7500. A furnishes twice as much as B, while C furnishes \$1500. How much does A furnish, and how much does B furnish?

18. A boy bought a bat, a ball, and a glove for \$2.25. If the bat cost twice as much as the ball, and the glove three times as much as the bat, what was the cost of each?

19. The sum of the angles of any triangle is 180°. In the triangle ABC the angle at A is three times as large as the angle at B, while the angle at C is twice that at A. What is the number of degrees in each?

FIG. 6.

20. A certain rectangle is four times as long as it is wide. The distance around it is 200 rods. Find its dimensions (length and breadth).

[HINT. The opposite sides of a rectangle are equal.]

13. Squares and Cubes. The product of two equal numbers is called the ***square*** of that number. Thus, $x \cdot x = x^2$, and is read "x square," just as in arithmetic $3 \times 3 = 3^2$ is read "three square."

The product of three equal numbers is called the ***cube*** of that number. For example, $x \cdot x \cdot x = x^3$, and is read "x cube."

14. Square Root and Cube Root. If there are two equal factors of a number, either is called the ***square root*** of the number. Thus, 25 has two equal factors, 5 and 5; hence the square root of 25 is 5. This is written $\sqrt{25} = 5$. In the same way, $\sqrt{x^2} = x$.

Similarly, if there are three equal factors of a number, any one of them is called the ***cube root*** of that number. Thus, 64 has three equal factors 4, 4, and 4; hence the cube root of 64 is 4. This is written $\sqrt[3]{64} = 4$. In the same way, $\sqrt[3]{x^3} = x$.

ORAL EXERCISES

Read each of the following expressions.

1. x^2+x^3.
2. $x-\sqrt{x}$.
3. $y^3+\sqrt[3]{x}$.
4. $a^2+a^3+\sqrt{a}$.
5. $\sqrt{m}+\sqrt[3]{m}+c^2$.
6. $4\sqrt[3]{z}+2\sqrt{z}$.
7. $a\sqrt{x}+b\sqrt{y}$.
8. gx^2+hx+i.
9. $p\sqrt[3]{y}-q\sqrt[3]{r}+1$.

15. Order of Operations. Operations are performed in the following order:

First, all *multiplications* and *divisions* in their order from left to right.

Second, all *additions* and *subtractions* in their order from left to right.

Thus, $6+8\cdot 3$, means six plus the product of eight and three; that is, $6+8\cdot 3=6+24=30$.

EXAMPLE 1. Find the value of $3\cdot 2+4\cdot 3-\frac{6}{2}+2^2$.

SOLUTION. $3\cdot 2+4\cdot 3-\frac{6}{2}+2^2=6+12-3+4=19$. *Ans.*

EXAMPLE 2. If $x=2$, $y=3$, $z=4$, find the value of

$$5\,xy+6\,z^2-\frac{yz}{3}.$$

SOLUTION. Giving x, y, and z their values, we have

$$\begin{aligned}5\cdot 2\cdot 3+6\cdot 4^2-\frac{3\cdot 4}{3}&=5\cdot 2\cdot 3+6\cdot 16-4\\&=30+96-4=122. \quad \textit{Ans.}\end{aligned}$$

ORAL EXERCISES

State the value of each of the following expressions.

1. $6+8\cdot 2-7$.
2. $10-3\cdot 2+\frac{12}{6}$.
3. $\frac{12}{3}-\frac{4}{2}+10$.
4. $20-3\cdot 4+6+3+6$.
5. $\frac{6\cdot 5}{3}-\frac{10}{5}+8$.
6. $6^2\cdot 2-6\cdot 3+\frac{16}{4}$.
7. $2^2\cdot 3^2-3\cdot 2^2$.
8. $10+2+\frac{8}{4}-\frac{20}{5}$.
9. $3\cdot 4^2-4\cdot 2+1$.
10. $2^3+3\cdot 4\cdot 2-3\cdot 5\cdot 2$

11. $\sqrt{25}+\frac{\sqrt{9}}{3}+\frac{2\sqrt{16}}{8}$.

12. $\frac{\sqrt[3]{8}}{2}-1+\frac{3\sqrt{9}}{27}$.

WRITTEN EXERCISES

If $a=1$, $b=2$, $c=3$, and $d=4$, find the value of each of the following expressions.

1. $5b-2c$.
2. $4c+3a$.
3. $d-2b$.
4. $c+2d$.
5. $d+b-2c$.
6. $3c-3a$.
7. $abcd-3bc$.
8. $\frac{2b+d}{2a}-2a$.

[Hint. In solving Ex. 8, remember that the numerator, $2b+d$, must be found first, then divided by $2a$.]

9. $\frac{2b+c-d}{c}+b$.
10. $\frac{3b-4a+10c}{2d}$.
11. $\frac{4c+5b-a}{d+c}$.
12. $\frac{b+2c+4bc}{b+2d+3b^2}$.
13. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.
14. $\frac{ab}{c}+\frac{bc}{a}+\frac{cd}{b}$.
15. $6d^2-4c+5b^2+1$.
16. $a^2+b^2+c^2+d^2$.
17. $4ab^2+6bc^2-2cd^2$.
18. $8ab+\sqrt{d}-d^2$.
19. $\frac{d}{2a+3b+c}$.
20. $\frac{\sqrt{d}+b^2+a^3}{c^2-2}$.
21. $\sqrt[3]{bd}+\sqrt{ad}$.
22. $\frac{\sqrt[3]{abd}+a^2b^2c^3}{b+\sqrt{d}-3}$.
23. $\frac{a^2b\sqrt{4a}+6\sqrt[3]{8a}}{c^2-b}+4a$.

24. Write in words the meaning of the expressions in exercises 1 to 8. For example, Ex. 1 means "five times b minus two times c."

EXERCISES — REVIEW OF CHAPTER I

I. Oral Exercises

1. If k represents a certain number, what represents 12 times that number?

2. If a yard of cloth costs m dollars, what will x yards cost?

3. If a boy rides b miles an hour, how far will he ride in c hours?

4. If a train goes y miles per hour, how fast does it travel per minute?

5. In how many hours can a man walk x miles if he goes at the rate of y miles per hour?

6. A man has a dollars and b quarters; how many cents has he?

7. How many dimes are there in x dollars and y half-dollars?

8. I have x dollars and y dimes. If I spend 60 cents, how much (in cents) have I left?

9. If y represents a certain number, what represents 6 less than three times y?

10. By how much does 25 exceed x?

11. If n oranges cost y cents, what is the price per dozen?

12. What is the cost of 2 dozen oranges at n cents per dozen and 3 dozen lemons at m cents per dozen?

13. If a yard be divided into x equal parts, how many inches will there be in each part?

14. If l stands for the length in feet of a running track, what is the length (in feet) of a track 50 feet longer? What is the length of a track 100 *yards* shorter?

15. If I am x years old now, how old was I y years ago? How old will I be c years hence?

16. A man sold an automobile for \$500 and lost x dollars. What did the automobile cost?

17. It costs 3 cents for each ounce or fraction thereof to send a letter through the mail. How much does it cost to send a letter that weighs a fraction more than n ounces?

II. Written Exercises

18. A rule stated in letters is called a ***formula***. For example, the area of a triangle equals one half the product of the base by the altitude. Stated as a formula, this becomes

$$A = \tfrac{1}{2}bh,$$

where A stands for *area*, b for *base*, and h for *altitude* (or height).

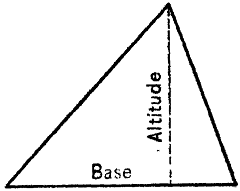


Fig. 7.

By use of this formula find the area of the triangle whose base is 6 inches and whose altitude is 3 inches.

[Hint. Substitute (or place) the given values of b and h into the formula and see what value results for A.]

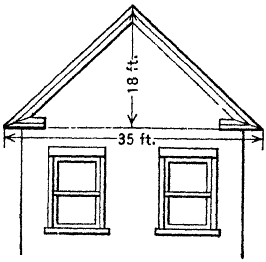


Fig. 8.

19. The gable of a certain house is a triangle whose base measures 35 feet and whose altitude is 18 feet. Find, by the formula of Ex. 18, how many square feet of lumber it contains.

20. It is shown in Geometry that "the square drawn on the hypotenuse of a right triangle is equal to the sum of the squares drawn on the other two sides." Express this rule in a formula, using h for hypotenuse, x for one side, and y for the other side.

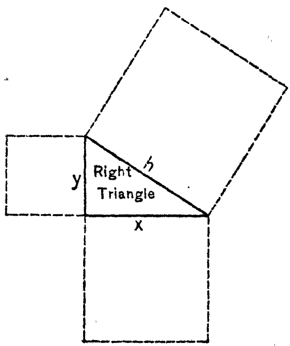


Fig. 9.

21. Find the area of the square on the hypotenuse of a right triangle if the sides are 6 feet and 10 feet in length; if the sides are 15 inches and 22 inches in length.

[Hint. Use the formula you obtained in Ex. 20.]

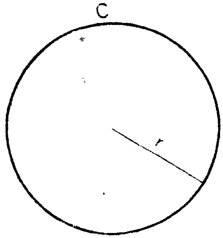


FIG. 10.

22. The circumference of a circle is expressed by the formula

$$C = 2\,\pi r,$$

where C stands for *circumference*, r for *radius*, and π (read pi) stands for the number 3.1416 (usually taken as $3\frac{1}{7}$).

By means of this formula find the circumference of the circle whose radius is 8 inches. Do the same when the radius is $\frac{3}{4}$ of an inch, when it is $2\frac{1}{3}$ feet, and finally when the *diameter* is 4 feet.

23. If my bicycle wheel has a diameter of 24 inches, how far does the bicycle go in one turn of the wheel?

[HINT. The distance moved = the length of the circumference.]

24. The formula for the radius of a circle is

$$r = \frac{C}{2\,\pi},$$

where r stands for the *radius*, C for the *circumference*, and π has the value mentioned in Ex. 22. Write out (in words) the rule thus expressed.

25. The formula for the area of a circle is $A = \pi r^2$, where r stands for the radius, A for the *area*, and π for the same number as in Ex. 22.

Write (in words) the rule thus expressed and find the area of the circle whose radius is 3 inches. Find the area also of the circle whose *diameter* is 28 inches.

26. A farmer builds a cylindrical silo which has for its base a circle 24 feet in diameter. How many square feet in the base? In working this, first use the value $3\frac{1}{7}$ for π, then use the more accurate value 3.1416. By how much do your two results differ?

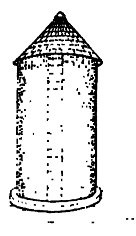


FIG. 11. — SILO.

27. The volume of a cylinder is the product of its height and the area of its base. Find the volume of the entire silo of Ex. 26, if its height is 50 ft.

Find the amount of material in the silo when it is full to a height of 15 ft.; 20 ft.; x ft.

28. The volume of a sphere is expressed by the formula

$$V = \tfrac{4}{3}\pi r^3,$$

where r stands for the radius (AO in Fig. 12), V for the volume, and π for the same number as in Ex. 22. By means of this formula, find the volume of a sphere whose radius is 3 inches.

Find the volume of a sphere whose radius is 4 feet.

Find the volume of a sphere whose *diameter* is 9 feet.

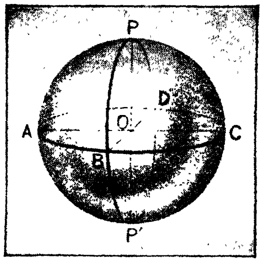


FIG. 12.

29. By the "gear" of a bicycle is meant the diameter of one of the wheels multiplied by the number of teeth in the front sprocket wheel divided by the number of teeth in the rear sprocket wheel. Express this rule in a formula, using the letter g for gear, D for diameter of wheel, T for number of teeth in front sprocket wheel, and t for number of teeth in rear sprocket wheel.

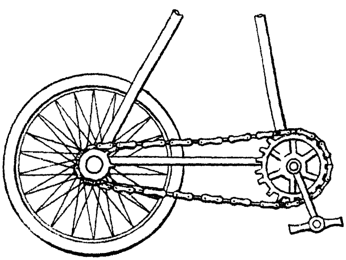


FIG. 13.

30. Use the formula you found in Ex. 29 to find the gear of the bicycle whose wheel measures 28 inches in diameter and whose front and rear sprocket wheels have 18 and 7 teeth, respectively. Try some other combinations to show the effect on the gear of changes in the number of teeth on the sprocket wheels.

31. In Fig. 14, one wheel is turning another by means of a belt.

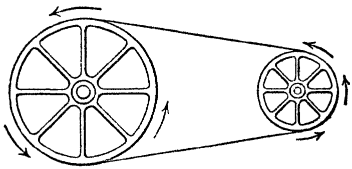


Fig. 14.

If we set

D = the diameter of the large wheel,

N = the number of turns it is making per minute,

d = the diameter of the smaller wheel,

n = the number of turns it is making per minute,

then during the motion we always have

$$DN = dn.$$

By means of this formula answer the following question: If the larger wheel is 1 foot in diameter and is making 10 turns (revolutions) per minute, how many turns per minute is the smaller wheel making if its diameter is 3 inches?

32. In the figure an engine is turning the central part (armature) of a dynamo. The armature wheel (which the belt runs

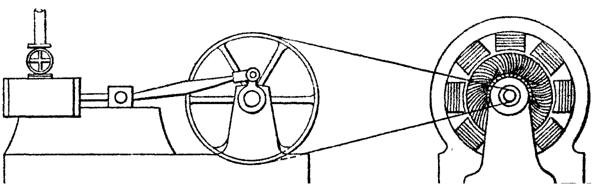


Fig. 15.

over) is 1 foot in diameter, while the engine wheel (driving wheel) is 6 feet in diameter.

How many revolutions must the driving wheel have per minute in order that the armature may revolve 1200 times per minute?

III. The Equation

Solve each of the following exercises by algebra; that is, let x represent the unknown number, form an equation, and solve it. Check each answer.

33. A farmer sold a horse and cow for \$210 He sold the horse for four times as much as the cow. How much did he get for each? \$42, \$168. *Ans.*

[Hint. Let x = the selling price of the cow.]

34. A plumber and two helpers get \$16.50 per day How much does each earn per day if the plumber earns four times as much as each helper?

35. In a business enterprise the combined capital of A, B, and C was \$8400.00. If A's capital was twice B's, and B's was twice C's, what was the capital of each?

36. Three newsboys sold a total of 60 papers. If the first sold twice as many as the second, and the third sold three times as many as the second, how many did each sell?

37. A and B began business with a capital of \$7500. If A furnished half as much as B, how much did each furnish?

[Hint. Let x = the number of dollars A furnished.]

38. Separate 72 into two parts one of which shall be one third the other.

39. A base ball nine won 12 games, which was three fourths of all the games it played. How many games did it play?

40. If one fifth of a certain number is added to the number the result is 12. What is the number?

41. The difference between two thirds of a certain number and two fifths of the same number is 16. What is the number?

42. A man pays a debt of \$91 with ten-dollar bills and one-dollar bills, paying three times as many one-dollar bills as ten-dollar bills. How many bills of each kind does he pay out?

43. A cablegram from New York to London costs 25 cents per word; one to Rome costs 31 cents per word; while one to Tokio

costs $1.33 per word. If a business man wishes to send the same message to each of the three cities and keep his total expense within $10, how many words may be in the message? Just how much will he spend?

44. A man has $100 in one bank and $25 in another. If he has $125 more to deposit, how should he divide it between the two banks so that the first account may become equal to the second?

[HINT. Let x = the amount to be deposited in the first bank. Then, $125-x$ will equal the amount to be deposited in the second bank.]

For further review exercises on this chapter, see Appendix, pp. 289-291.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

16. Introduction. In Figure 16 are two boys having a "tug of war" at the ends of a rope. One boy is pulling with a force of 50 lb., while the other boy is pulling with a force of 45 lb. This can be described very briefly by saying that one boy is pulling with a force of $+50$ lb. (read *plus* 50 lb.) while the other boy is pulling with a force of -45 lb. (read *minus* 45 lb.). The two signs $+$ and $-$ as thus used simply indicate that the two pulls are *opposite* in direction.

In just the same way the $+$ and $-$ signs may be used in reading a thermometer. Thus, $+10°$ indicates $10°$ *above* zero, while $-10°$ represents $10°$ *below* zero.

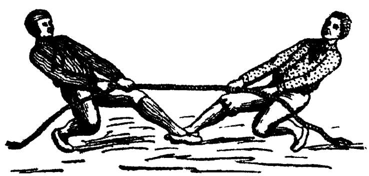


FIG. 16.

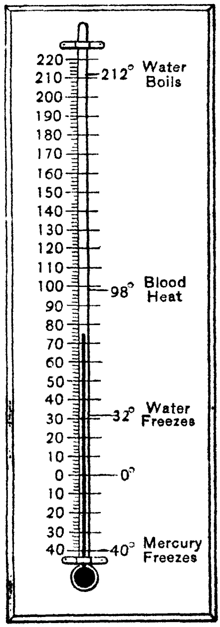


FIG. 17.

Other familiar illustrations are the following: $+26°$ of latitude means $26°$ *north* latitude, while $-26°$ latitude

means 26° *south* latitude. Again, a business man may use +\$15 to represent an *asset* (amount due) of \$15, and he may use −\$15 to represent a *liability* (amount owed) of \$15. Thus, to say that you have −\$10 means that you *owe* that amount.

Many other illustrations of this idea might be given, but these are enough to make clear how we often meet with quantities which seem to have precisely opposite senses. Numbers dealing with such quantities are called *opposite* numbers. The one given the + sign is called ***positive,*** while the one given the − sign is called ***negative.***

ORAL EXERCISES

1. Using Fig. 17, give with proper sign the number which represents the boiling point. Do the same for the freezing point of water, for the freezing point of mercury, and for the temperature called *blood heat.*

2. What sign should be used to indicate the latitude of New York City? Of St. Louis? Of Buenos Ayres? Of Calcutta?

3. If we use the + sign to denote a man's income, what will the − sign denote?

4. Express by means of suitable signs the following: \$18 gain; \$20 loss; a \$40 debt; a liability of \$100; an asset of \$150; \$8 profit.

5. Augustus Cæsar was born in the year 63 B.C. and died in the year 14 A.D. What do these dates become if we use the + sign for years that are A.D.?

6. What is meant by saying that the Pyramids of Egypt were built about the year −2500?

7. State what each of the following means when the number in it has its sign changed to −.

(*a*) John has $2.

SOLUTION. If we change the 2 to -2, the statement becomes "John has $-$ $2," which means "John owes $2."

(*b*) My home is 25 miles northwest.
(*c*) They won the game by 8 points.
(*d*) The clock is 1 hour fast.
(*e*) I overslept 2 hours.
(*f*) An iron rod expanded 3 inches.
(*g*) The rabbit ran away from the hunter at the rate of 15 miles an hour.
(*h*) John owes $2.
(*i*) They lost the game by 8 points.
(*j*) The population of the town increased by 230.
(*k*) The rock extends 3 feet underground.
(*l*) The wind carried us 8 miles toward shore.
(*m*) He took 4 oranges out of the basket.
(*n*) She added 100 words to her essay.
(*o*) He turned on 10 lights.
(*p*) After the bell rang, 5 people entered the room.
(*q*) Please open the door about a foot more.
(*r*) School begins at 8 o'clock.

For further exercises see the review list, p. 47, *and Appendix, p.* 291.

17. Addition of Positive and Negative Numbers. If you have $10 and receive $6 more, you have $16. This may be expressed by writing

$$(+10)+(+6)=+16.$$

Again, if you have $10 but owe $8, what you really have is $2. This may be expressed by writing

$$(+10)+(-8)=+2.$$

Again, if you have \$10 but owe \$11, what you really have is a debt of \$1. This may be expressed by writing

$$(+10)+(-11)=-1.$$

Finally, if you owe \$10 and for some reason are obliged to owe \$6 more, what you really have is a debt of \$16. This may be expressed by writing

$$(-10)+(-6)=-16.$$

These four illustrations when examined carefully show that the rule for adding any two numbers (positive or negative) is as follows.

RULE FOR ADDING TWO NUMBERS. *To find the sum of two numbers whose signs are* ***opposite****, take their difference, regarding each as positive, and prefix the sign of the larger number to the result.*

To find the sum of two numbers whose signs are the ***same****, take their sum, regarding each as positive, and prefix the common (same) sign to the result.*

NOTE. Just as it is the stronger boy who wins in the tug of war (see Fig. 16, § 16), so, in adding two numbers whose signs are opposite, the stronger, or larger, number is the one that leaves its sign upon the answer. For example, $(+16)+(-8)=+8$, but $(-16)+(+8)=-8$.

ORAL EXERCISES

State the sum in each of the following exercises, and explain how your answer comes from the Rule above.

1. A gain of \$8 and a gain of \$4.
2. A gain of \$10 and a loss of \$3.
3. A loss of \$5 and a gain of \$15.
4. A loss of \$10 and a loss of \$15.
5. A debt of \$60 and an asset of \$80.
6. A rise in temperature of 10° and a fall of 20°.

Perform the additions in each of the following exercises.

7. $\begin{array}{r} +10 \\ +6 \\ \hline \end{array}$ **8.** $\begin{array}{r} -10 \\ -6 \\ \hline \end{array}$ **9.** $\begin{array}{r} +10 \\ -6 \\ \hline \end{array}$ **10.** $\begin{array}{r} -10 \\ +6 \\ \hline \end{array}$

11. $\begin{array}{r} +125 \\ -75 \\ \hline \end{array}$ **12.** $\begin{array}{r} -70 \\ +8 \\ \hline \end{array}$ **13.** $\begin{array}{r} +63 \\ +12 \\ \hline \end{array}$ **14.** $\begin{array}{r} -75.2 \\ -2.2 \\ \hline \end{array}$

15. $\begin{array}{r} -25 \\ +70 \\ \hline \end{array}$ **16.** $\begin{array}{r} +30 \\ -62 \\ \hline \end{array}$ **17.** $\begin{array}{r} +125 \\ -136 \\ \hline \end{array}$ **18.** $\begin{array}{r} -73 \\ +126 \\ \hline \end{array}$

19. Two boys start from the same place and walk in opposite directions, the one going east at the rate of 3 miles an hour and the other going west at the rate of 2 miles an hour. Taking the east direction as positive, state (by a number) the position of each at the end of two hours.

20. A boat was rowed upstream 2 miles and then allowed to float 5 miles downstream. Taking upstream as positive, what number represents the position of the boat at the end of the trip?

18. Addition of Several Numbers. When we add several numbers instead of merely two, we first add the + numbers by themselves, then the − numbers by themselves, then we add (by the rule in § 17) the two sums thus obtained.

For example, in finding $(+8)+(-7)+(-6)+(+4)$ the steps are as follows:

$$\begin{array}{r} +8 \\ +4 \\ \hline +12 \end{array} \qquad \begin{array}{r} -7 \\ -6 \\ \hline -13 \end{array} \qquad (+12)+(-13)=-1. \quad \textit{Ans.}$$

19. Note. Instead of $+5$ it is customary to write simply 5, and in the same way $+3$ is written simply 3, and so on. In other words, whenever a number occurs *without* any sign, the + sign is to be understood. The − sign is never omitted.

ORAL EXERCISES

State the sum in each of the following exercises.

1. $(+3)+(-4)+(+5)$.
2. $2+(-1)+4$.

   [HINT. See § 19.]
3. $(-4)+6+(-1)$.
4. $(-3)+(-4)+3+2$.
5. $6+7+3+(-6)$.
6. $2+(-1)+3+1+(-2)$.
7. $(-2)+(-3)+(-1)+2$.
8. $(-2)+3+(-4)+7$.
9. $(-1)+2+(-3)+1$.
10. $16+(-15)+4$.
11. $(-21)+3+7+(-1)$.

WRITTEN EXERCISES

Find the sum in the following exercises.

1. 15, -12, -32, 8, and -4.
2. -17, 22, -6, -4, and 2.
3. 18, 20, -2, -18, and -6.
4. -4, 7, 8, -8, and -7.
5. 24, 6, 5, -10, and -7.
6. -4, -7, -3, -1, and 6.
7. -7, -8, 2, 3, and -4.
8. 4, -7, -1, -6, and 10.

9. At the beginning of the year a class in algebra had 30 members; during the year 4 entered and 6 withdrew. How many were in the class at the end of the year? Show how to obtain your answer by using a negative number to represent those who withdrew.

10. By use of negative numbers, solve the following example:

If a carrier pigeon can fly 60 miles an hour, at what rate will it go when flying directly against a wind blowing at the rate of 40 miles an hour?

[HINT. Give each rate its proper sign and add.]

11. Solve Ex. 10 when it is supposed that the pigeon can fly only 30 miles an hour. What does the negative sign of your answer indicate?

12. A steamer which can go 10 miles an hour in still water is running against a current flowing 8 miles an hour. How fast and in what direction is the steamer moving. Work by negative numbers.

13. Solve Ex. 12 when the steamer is not only going at 10 miles an hour against a current of 8 miles an hour, but a wind is blowing against it at the rate of 1 mile an hour.

14. An elevator starts from a certain floor, goes up 50 feet, then down 30 feet, up 35 feet, up 45 feet, and there stops. What number represents its final position with reference to the starting place?

15. Alexander the Great founded the city of Alexandria in Egypt in the year -322, and started there a great university. This university lasted for 292 years, when it fell into the hands of the Romans, who continued it for 671 years more. In what year did it close?

16. Add $3a$, $2a$, and $-4a$. [HINT. See § 11.]

17. Add xy, $-3xy$, and $4xy$.

18. Find the value of each of the following when $x=1$ and $y=2$. (See § 15.)

(*a*) $-x+5$.

(*b*) $-x+y$.

(*c*) $x+(-y)$.

(*d*) $-x^2+y+y$.

(*e*) $-4x^3+2y-x$.

(*f*) $\dfrac{-2y+5}{x}$.

(*g*) $\sqrt{2y}+(-x)+x^2$.

(*h*) $-x+\sqrt{2y}-y+y^2$.

(*i*) $-\sqrt[3]{2y^2}+\sqrt{2y}+1$.

19. Tell (by inspection) what value x must have in each of the following equations.

(*a*) $x+(-1)=1$.

(*b*) $x+1=-1$.

(*c*) $x+2=-3$.

(*d*) $2+x=1$.

(*e*) $-3+x=-1$.

(*f*) $x+1=3$.

20. The Size of Numbers. In arithmetic we think of 3 as greater than 2, of 5 as greater than 3, of 4 as less than 7, etc. Also, we regard 0 (called *zero*) as the least of all numbers. However, when we come to use negative numbers as well as positive, as we have been doing in this chapter, we must regard -1 as even less than 0, for if a man has a *debt* of \$1 (that is, if he has $-\$1$) he really has that much less than no money at all. Likewise, -2 we must regard as less than 0, and indeed it must be less than -1. In the same way, -5 is less than -3, while -10 is less than -8, etc.

The whole situation in this matter is vividly brought out to the eye in the figure below: Here the $+$ numbers are arranged in their order (as on a yardstick) running

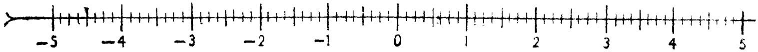


Fig. 18.

to the *right* of the point marked 0, while the $-$ numbers are similarly arranged to the *left* of that point. Observe that if you start anywhere on the line and go to the *right*, the numbers you meet are constantly *increasing* (in the sense explained above), while if you go to the *left*, they are constantly *decreasing*. Thus the figure shows *all* numbers (positive and negative) arranged in their increasing order as one reads from left to right.

In the figure above only the *integers*, 1, 2, 3, etc. and their negatives, -1, -2, -3, etc. are printed in, but a complete figure would give markings also for fractions such as $\frac{1}{2}$, $2\frac{1}{3}$, $-\frac{3}{4}$, $-5\frac{7}{8}$. Thus, $\frac{1}{2}$ goes at the point situated halfway between 0 and 1; again, $2\frac{1}{3}$ goes at the point one third the way from 2 to 3; $-\frac{3}{4}$ goes at the point three fourths the way from 0 to -1, and $-5\frac{7}{8}$ goes at the point which is seven eighths the way from -5 to -6. In this way, *every* fraction has a definite position on the line.

ORAL EXERCISES

In each of the following exercises, state which of the two numbers is the larger. To do this, locate the numbers at their proper places on the line shown in the figure on page 34; or better, on a similar line which you can now draw for yourself.

1. $6, 9.$
2. $6, -9.$
3. $-6, 9.$
4. $-6, -9.$
5. $\frac{2}{3}, \frac{3}{4}.$
6. $\frac{2}{3}, -\frac{3}{4}.$
7. $-\frac{2}{3}, \frac{3}{4}.$
8. $-\frac{2}{3}, -\frac{3}{4}.$
9. $2\frac{1}{2}, 1\frac{7}{8}.$
10. $-2\frac{1}{2}, -1\frac{7}{8}.$
11. $-\frac{7}{9}, 0.$
12. $-.8, -6.3$

21. Subtraction of Positive and Negative Numbers. Suppose you owe \$2, but your father, without your knowing it, pays the debt. The result is that you are \$2 better off than before. In other words, the result of taking away, or subtracting, a \$2 debt is to *increase* what you have by \$2. If, for example, you had \$10 in cash before the debt was paid, all you *really* had was \$8, but now the whole \$10 is yours. Stated in terms of negative numbers, this means that subtracting -2 from 8 is the same as adding 2 to 8; that is,

$$8-(-2)=8+2=10.$$

Now, let us suppose another case. Suppose you are \$2 in debt when you accidentally break a window valued at \$5. Even though you have no money to pay for it at the moment, you really have had \$5 taken away, or subtracted from you. It is just the same as adding a new \$5 debt to the old \$2 one, and the result is a \$7 debt. In terms of

negative numbers, this says that subtracting 5 from -2 is the same as adding -5 to -2; that is,

$$-2-5=-2+(-5)=-7.$$

Next, suppose you have $10, but wish to buy a picture costing $15. This is like taking away, or subtracting, $15 from yourself when you have only $10, and the result is that you go $5 into debt. It is the same as adding a $15 debt to the $10 that is in your pocket; that is,

$$10-15=10+(-15)=-5.$$

Finally, suppose you owe $10, but your father pays $8 of this amount. You then owe but $2. This is really subtracting an $8 debt from a $10 debt, and its effect is the same as though your father had made you a present of $8, simply adding it to whatever you had in the first place. Stated in terms of negative numbers, this says that

$$-10-(-8)=-10+8=-2.$$

A careful study of the four cases above shows us two important things:

(1) When both positive and negative numbers are being used, we may always subtract one number from another, no matter whether the number subtracted is smaller than the one from which it is taken (as in arithmetic) or not. For example, in the third case we subtracted 15 from 10, obtaining the result -5.

(2) The rule for subtracting any number (positive or negative) from another is as follows.

RULE FOR SUBTRACTING ONE NUMBER FROM ANOTHER. *Change the sign of the subtrahend (or number subtracted) and proceed as in addition.* (See § 17.)

For example, to subtract -6 from 3 we simply change the -6 to 6 and add the result to 3; that is, we have

$$3-(-6)=3+6=9. \quad \textit{Ans.}$$

ORAL EXERCISES

1. State the result of each of the following subtractions.

(*a*) $10-(-6)$. (*c*) $3-6$. (*e*) $5a-6a$.
(*b*) $-10-(-6)$. (*d*) $-4-(-6)$. (*f*) $-4x-(-5x)$.

2. In each of the following exercises subtract the smaller number from the larger one.

(*a*) 2, 3. (*e*) 2, -3. (*i*) $-\frac{1}{2}$, $\frac{1}{4}$.
(*b*) -1, 2. (*f*) 1, -1. (*j*) $-\frac{1}{2}$, $-\frac{1}{4}$.
(*c*) 6, -7. (*g*) $\frac{1}{2}$, $\frac{1}{4}$. (*k*) -1, $-\frac{2}{3}$.
(*d*) -5, -6. (*h*) $\frac{1}{2}$, $-\frac{1}{4}$. (*l*) $-\frac{3}{4}$, -2.

3. In each of the following exercises subtract the larger number from the smaller one.

(*a*) 6, 8. (*e*) -2, -3. (*i*) $-\frac{1}{2}$, $\frac{1}{4}$.
(*b*) -1, 2. (*f*) 2, -3. (*j*) $-\frac{1}{2}$, $-\frac{1}{4}$.
(*c*) 4, 1. (*g*) $\frac{1}{2}$, $\frac{1}{4}$. (*k*) -3, $-\frac{3}{4}$.
(*d*) 4, -1. (*h*) $\frac{1}{2}$, $-\frac{1}{4}$. (*l*) $-\frac{2}{3}$, 4.

4. On a certain day the thermometer stood at 75°. The next day it stood at 45°. What was the drop in temperature?

5. On a certain day the thermometer stood at 10°. The next day it stood at $-6°$. What was the drop in temperature?

[HINT. As in Ex. 4, subtract the lower temperature from the higher.]

6. How long was it from the year $+60$ to the year $+100$?

7. Augustus Cæsar lived from the year -63 to the year $+14$. How old was he when he died?

[HINT. As in Ex. 6, subtract the earlier date from the later one.]

8. One ship had a latitude of $+25°$, while another one had a latitude of $-18°$. What was their difference in latitude?

WRITTEN EXERCISES

1. Find the result of each of the following subtractions.

(*a*) $192-261$. (*d*) $80-(-45)$. (*g*) $.6-.7$
(*b*) $-150-270$. (*e*) $\frac{1}{2}-\frac{1}{3}$. (*h*) $2.5-4.9$
(*c*) $80-45$. (*f*) $0-\frac{1}{2}$. (*i*) $2.5-4.09$

2. When ready to ascend, a balloon, including its basket, weighed -1500 lb. If the basket alone weighed 100 lb., how much did the balloon alone weigh?

3. The weather map for February 1, 1916, gave the following as the maximum (highest) and minimum (lowest) temperatures for that day:

	Chicago	Duluth	Helena	Montreal	New Orleans	New York
Maximum	30°	3°	− 5°	18°	75°	41°
Minimum	24°	−7°	−13°	−12°	63°	21°

From this table it appears that the *range* of temperature at Chicago was $30°-24°$, or $6°$. What was the range at each of the other cities mentioned in the table?

4. From the table in Ex. 3, calculate how far below the freezing point (which is 32°) the temperature fell in Montreal.

5. Referring again to the table in Ex. 3, how much colder did it become in Duluth than in Chicago? in Montreal than in New York? in Helena than in New Orleans?

6. If $x=1$ and $y=2$, find the value of each of the following expressions.

(*a*) $2x-3y$. (*d*) $\sqrt{2y}-x$. (*g*) $\sqrt[3]{4y}-x^2$.
(*b*) x^2-y^2. (*e*) $\sqrt{2y}-(-x)$. (*h*) $\sqrt[3]{6y-4x}$.
(*c*) x^3-y^3. (*f*) $-\sqrt{2y}-(-x)$. (*i*) $\sqrt[3]{6y-(-15)}$.

7. Tell (by inspection) what must be the value of x in each of the following equations.

(*a*) $x-1=2$. (*c*) $2+x=1$. (*e*) $-3+x=-1$.
(*b*) $x-2=-3$. (*d*) $2+x=-1$. (*f*) $x-4=-5$.

8. What is the number which added to 2 gives -4?

[HINT. Let x represent the unknown number, form an equation, and then solve it by inspection. Compare § 6.]

For further exercises on this topic, see review exercises, p. 47, and Appendix, p. 292.

22. Addition and Subtraction of Several Numbers. In adding and subtracting several numbers we may proceed from left to right, performing each addition and subtraction as we come to it. For example,

$$5+3-7-10+1=8-7-10+1=1-10+1$$
$$=-9+1=-8. \quad \textit{Ans.}$$

NOTE. Another way is to add the + terms by themselves and the − terms by themselves, then take the sum of the two results. Thus, in the example above we have

$$\begin{array}{r} +5 \\ +3 \\ \underline{+1} \\ +9 \end{array} \qquad \begin{array}{r} -7 \\ \underline{-10} \\ -17 \end{array} \qquad 9+(-17)=-8. \quad \textit{Ans.}$$ (Compare with § 18.)

WRITTEN EXERCISES

Simplify each of the following expressions.

1. $2-6+4-3-2$. **2.** $-2+4-6+7+3$.

3. $1-2+3-4+5-6+7$. **4.** $1+(-2)-(-3)+4$.

5. $2-(-3)+(-4)-(-6)-7$. **6.** $25-30+17-21-45$.

7. $101-75+36-175-256$. **8.** $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}$.

9. $\frac{1}{5}-\frac{1}{4}+\frac{1}{3}-\frac{1}{2}$. **10.** $.2-.6+.08-.004$

11. $.001-.01+.1-1+10$. **12.** $2-\frac{1}{3}-3\frac{1}{4}+.2-.01$

13. Do you know of any game in which there are negative counts? Describe the method of counting. How do you find the total score?

14. Express as a continued addition the bank account of a man who (1) has \$67.50, (2) spends \$5.46, (3) deposits \$21.00, (4) spends \$41.25, (5) spends \$12.50, (6) deposits \$14.15, (7) spends \$7.20. What will his final bank account be?

23. Multiplication of Positive and Negative Numbers. In arithmetic the numbers which we multiply together are always positive, but in algebra some (or all) of the numbers to be multiplied may be negative. Four different cases are here possible and will now be illustrated.

(1) Consider the product 5×4. This means, as in arithmetic, that 5 is to be taken 4 times. Hence the result is 20; that is,

$$5\times 4=20.$$

(2) Consider the product $(-5)\times 4$. This means that -5 is to be taken 4 times. Hence, by the Rule in § 17, the result must be -20; that is,

$$(-5)\times 4=-20.$$

(3) Consider the product $5\times(-4)$. Remembering that in arithmetic we have $2\times 3=3\times 2$, we may here suppose that $5\times(-4)=(-4)\times 5$. But this last form (being similar to the one in the *second* case) means -4 taken 5 times, which is -20. Thus, we have

$$5\times(-4)=-20.$$

(4) Consider the product $(-5)\times(-4)$. We naturally look upon this as meaning the *negative* of $5\times(-4)$; that is, the negative of -20. But the negative (or opposite) of -20 is $+20$, or simply 20. Whence, we write

$$(-5)\times(-4)=20.$$

These four illustrations make clear the following rule.

RULE FOR MULTIPLICATION OF TWO NUMBERS. *Multiply as in arithmetic, prefixing the + sign to the product if the two numbers have the* **same** *sign, and prefixing the − sign to the product if the two numbers have* **opposite** *signs.*

ORAL EXERCISES

State the value of each of the following expressions.

1. 3×2.
2. $3\times(-2)$.
3. $(-3)\times2$.
4. $(-3)\times(-2)$.
5. $4\times(-7)$.
6. $(-10)\times(-6)$.
7. $a\times(-b)$.
8. $(-a)\times b$.
9. $(-a)\times(-b)$.
10. $(-\frac{2}{3})\times\frac{3}{8}$.
11. $(-\frac{1}{2})\times\frac{3}{8}$.
12. $(-\frac{1}{3})\times\frac{3}{4}$.
13. $(-\frac{1}{3})\times(-\frac{1}{4})$.
14. $(-\frac{2}{3})\times(-\frac{3}{2})$.
15. $3\times(-x^2)$.
16. $(-x^2)\times(-y^2)$.
17. $(-1)\times\left(-\frac{x^2}{y^2}\right)$.
18. $(-2+1)\times(3-4)$.
19. $\frac{a}{b}\times\left(-\frac{a}{b}\right)$.
20. $\left(-\frac{a}{b}\right)^2$.
21. $(-1)\left(-\frac{a}{b}\right)^2$.

24. Multiplication of Several Numbers. The product of three or more numbers is found by performing one multiplication at a time. Thus, in finding $6\times(-5)\times(-4)\times3$ the steps are as follows:

$$6\times(-5)=-30,$$
$$(-30)\times(-4)=120,$$
$$120\times3=360.$$

Therefore, $6\times(-5)\times(-4)\times3=360$. *Ans.*

NOTE 1. If the number of *negative* factors in a product is *even*, the sign of the product is +; but if the number of negative factors is *odd*, the sign of the product is −. That this is so, follows from the Rule given in § 23.

For example, the sign of $2\times(-3)\times(-4)\times5$ is +, since we here have two negative factors (or an *even* number); but the sign of $2\times(-3)\times(-4)\times5\times(-7)$ is −, since here we have three (or an *odd* number) of the negative factors.

NOTE 2. The sign $\times$ may be omitted in products containing *literal* numbers. Thus, instead of $2\times(-a)\times(-b)\times c$ we write $2(-a)(-b)c$. This is merely an extension of what was said in § 3.

ORAL EXERCISES

In each of the following, state first the *sign* of the result, then give the complete value.

1. $2\times(-5)\times(-8)$.
2. $4\times(-3)\times 2$.
3. $(-6)\times(-2)\times 5$.
4. $(-1)\times(-2)\times(-3)\times(-4)$.
5. $(-1)\times(-1)\times(-1)\times(-1)\times(-1)$.
6. $(-2)\times(-\frac{1}{2})\times 4$.
7. $\frac{1}{2}\times(-\frac{2}{3})\times(-\frac{3}{4})$.
8. $.1\times(-.2)\times(-.3)$.
9. $(-a)(-b)(-c)$. *Ans.* $-abc$.
   [HINT. See Note 2, in § 24.]
10. $(-2a)(-6b)(-2c)$.
11. $(-a)b(-4c)$.
12. $(-x)(-y)(-z)(-w)$.
13. $mn(-2rs)$.
14. $xyz(-abc)(-3pqn)$.
15. $3p(-2q)r^2$.
16. $x(-y^2)(-z^3)$.

WRITTEN EXERCISES

If $a=-1$, $b=-2$, and $c=-3$, find the value of each of the following expressions.

1. $2abc$.
2. $2(-a)bc$.
   [HINT. $-a=-(-1)=1$.]
3. $2(-a)(-b)c$.
4. $2(-a)(-b)(-c)$.
5. $2ab+3bc$.
6. $2ab-3bc$.
7. $ab+bc+ac$.
8. $ab+bc-ac$.
9. $ab-bc-ac$.
10. $a^2+b^2+c^2$.
11. $a^3+b^3+c^3$.
12. $\sqrt{-2b}$.
13. $\sqrt{2b-3c}$.
14. $\dfrac{2ab+c}{3c+b}$.
15. $\dfrac{abc+6}{4c+6b}$.
16. $a^3b^3c^3$.
17. $a^2b^2c^2-abc$.
18. ab^2+b^2c.
19. $\sqrt{4a^2b^2+c^2}$.
20. $\sqrt[3]{6c^2-5b}$.

For further exercises on this topic, see Appendix, p. 292.

25. Powers and Exponents. We have already noted (see § 13) that x^2 means $x \times x$ and may be read "x square." It may also be read "x to the *second* power." In the same way, x^3 means $x \times x \times x$ and may be read either "x cube" or "x to the *third* power."

We now note also that x^4 means $x \times x \times x \times x$ and is read "x to the *fourth* power." Again, x^5 means $x \times x \times x \times x \times x$ and is read "x to the fifth power," etc. In all these cases the number denoting the power is called the ***exponent.*** Thus, 2 is the exponent in x^2, while 3 is the exponent in x^3, etc.

26. Signs of the Powers of Numbers. In the first set below can be seen the powers of 2 from the first up to the sixth, while in the second set are the same powers of -2:

$$
\begin{aligned}
2^1 &= 2\\
2^2 &= 2 \times 2 = 4\\
2^3 &= 2 \times 2 \times 2 = 8\\
2^4 &= 2 \times 2 \times 2 \times 2 = 16\\
2^5 &= 2 \times 2 \times 2 \times 2 \times 2 = 32\\
2^6 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64
\end{aligned}
$$

$$
\begin{aligned}
(-2)^1 &= -2\\
(-2)^2 &= (-2) \times (-2) = +4\\
(-2)^3 &= (-2) \times (-2) \times (-2) = -8\\
(-2)^4 &= (-2) \times (-2) \times (-2) \times (-2) = +16\\
(-2)^5 &= (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -32\\
(-2)^6 &= (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) = +64
\end{aligned}
$$

Observe that the results in the first set are all positive, while those in the second set are positive or negative according as the exponent used is an even or an odd number. This illustrates the following rule.

Rule of Signs for Powers. *All powers of a positive number are positive. An* ***even*** *power of a negative number is positive, while an* ***odd*** *power of a negative number is negative.*

ORAL EXERCISES

In each of the following exercises, state first the *sign* of the result, then give the complete answer.

1. $(-1)^4$.	**8.** $(-6)^3$.	**15.** $(-a)^3$.
2. $(-4)^3$.	**9.** $(-1)^6$.	**16.** $(-x)^4$.
3. $(-1)^5$.	**10.** 3^3.	**17.** $(-y)^5$.
4. 1^5.	**11.** $(-3)^3$.	**18.** $(-1)^2\times(-2)^2$.
5. $(-6)^2$.	**12.** 3^4.	**19.** $(-a)^2(-b)^2$.
6. $(-1)^7$.	**13.** $(-3)^4$.	**20.** $xy(-z)^3$.
7. $(-5)^3$.	**14.** $(-a)^2$.	**21.** $(-m)^2(-n)^3(-p)^4$.

27. Division of Positive and Negative Numbers. To say that $4\times5=20$ is the same as saying that $20\div5=4$.

Likewise, to say that $4\times(-5)=-20$ is the same as saying that $(-20)\div4=-5$.

In the same way, from $(-4)\times5=-20$ we get $(-20)\div(-4)=5$, and finally from $(-4)\times(-5)=20$ we have $20\div(-4)=-5$.

These four cases illustrate all the possibilities in the division of one number (positive or negative) by another; from them we derive the following rule.

Rule for Division of Two Numbers. *Divide the dividend by the divisor as in arithmetic, giving the positive sign to the quotient if both dividend and divisor have the* ***same*** *sign, but giving the negative sign to the quotient if the dividend and divisor have* ***opposite*** *signs.*

ORAL EXERCISES

In each of the following exercises state first the *sign* of the result, then give the complete answer.

1. $25 \div 5$.
2. $(-27) \div 9$.
3. $(-27) \div (-9)$.
4. $27 \div (-9)$.
5. $(-25) \div (-5)$.
6. $30 \div (-6)$.
7. $(-72) \div 12$.
8. $(-60) \div (-4)$.
9. $(-10) \div 10$.
10. $(-8) \div 8$.
11. $90 \div (-15)$.
12. $(-63) \div (-7)$.
13. $(-100) \div 25$.
14. $121 \div 11$.
15. $(-144) \div (-12)$.
16. $12\,a \div 4$.
17. $12\,a \div (-4)$.
18. $(-12\,a) \div (-4)$.
19. $(-12\,b) \div (-6)$.
20. $(-8\,xy) \div 8$.
21. $(-144\,r) \div (-12)$.
22. $16\,abc \div 4$.
23. $\frac{1}{2} \div (-\frac{1}{2})$.
24. $(-\frac{3}{4}) \div (-\frac{1}{2})$.
25. $(-2.5) \div (2.5)$.
26. $7.26 \div (-2.42)$.
27. $(-a) \div b$; $-\frac{a}{b}$. *Ans.*
28. $a \div (-b)$.
29. $(-a) \div (-b)$.
30. $\frac{-n}{m}$.
31. $-4\,x \div x$.
32. $-4\,x \div (-x)$.

33. Give (by inspection) the value of x in each of the following equations.

(*a*) $-x = 6$.	(*e*) $-4\,x = -16$.	(*i*) $-4 = 2\,x$.
(*b*) $4\,x = -8$.	(*f*) $-6\,x = 36$.	(*j*) $-10 = 5\,x$.
(*c*) $-3\,x = -12$.	(*g*) $3\,x = -15$.	(*k*) $\frac{6}{x} = -3$.
(*d*) $-5\,x = 10$.	(*h*) $-2\,x = 13$.	(*l*) $\frac{-2}{x} = \frac{1}{2}$.

WRITTEN EXERCISES

If $a=2$, $b=-3$, and $c=-5$, find the value of each of the following expressions.

1. $\dfrac{6a}{b}$.
2. $\dfrac{2abc}{a}$.
3. $\dfrac{ab^2}{6}$.
4. $\dfrac{abc}{5}$.
5. $\dfrac{2ab}{4}$.
6. $\dfrac{4bc}{a^2}$.
7. $\dfrac{a+c}{b}$.
8. $\dfrac{a+b}{a-b}$.
9. $\dfrac{ab+ac-bc}{3c}$.
10. $\dfrac{5a-3b+2c}{3c}$.
11. $\dfrac{abc-3bc+c}{6a^2}$.
12. $\dfrac{a^2+b^2+c^2}{a+b+c}$.
13. $\dfrac{a}{b}\times\dfrac{a}{c}$.
14. $\dfrac{\sqrt{2a}-bc}{\sqrt{bc+1}}$.
15. $\dfrac{a-b}{a+b}\times\dfrac{b-c}{b+c}$.
16. $\dfrac{a^3+b^3+c^3}{a^2+b^2+c^2}$.
17. $\dfrac{a^4+b^4+c^4}{a+b+c}$.
18. $\dfrac{a^2+b^2}{a+b}\times\dfrac{a^4+b^4}{a^3+b^3}$.
19. $abc\times\dfrac{a+b}{b+c}$.
20. $abc\div\dfrac{a+b}{b+c}$.

21. Explain how the equation $(-4)\div(-2)=2$ illustrates the following statement: "The boy who owes \$4 owes twice as much as the boy who owes \$2."

22. Taking the equation in Ex. 21, write out the statement it illustrates with reference to the latitudes of two ships. [HINT. See § 16.]

23. Write (using negative numbers whenever necessary) the equation corresponding to the following statement: "If John owes \$10 and I owe half as much, I owe \$5."

24. Explain how the equation $10\div(-2)=-5$ illustrates the following statement: "If I have \$10 and you owe \$2 then I have five times as much as you *but in the opposite sense*."

For further exercises on this topic, see the review exercises, p. 47, and Appendix, p. 293.

EXERCISES — REVIEW OF CHAPTER II

1. The words "good" and "bad" have precisely opposite meanings; hence the one may be called the *negative* of the other. Two such words are also called *antonyms* of each other. State what is the negative (or antonym) of each of the following expressions.

(*a*) slow	(*e*) to find anything	(*i*) darkness
(*b*) rich	(*f*) to give away something	(*j*) evil
(*c*) difficult	(*g*) to go to sleep	(*k*) dwarf
(*d*) industrious	(*h*) to stand up	(*l*) doubt

2. What tense (in grammar) is the negative of the past tense?

3. What word describes the negative of "good health"?

4. If you have any two numbers (positive or negative) how do you decide which is the greater?

[HINT. See § 20.]

5. In each of the following cases, subtract (mentally) the smaller number from the larger.

(*a*) $5, 6$.	(*c*) $14, -10$.	(*e*) $\frac{1}{3}, \frac{2}{3}$.	(*g*) $.5, -.6$
(*b*) $-1, 3$.	(*d*) $-4, -5$.	(*f*) $\frac{1}{3}, -\frac{2}{3}$.	(*h*) $-\frac{1}{2}, .3$

6. In each of the examples in Ex. 5 subtract (mentally) the larger number from the smaller.

7. Find (mentally) the value of each of the following expressions.

(*a*) $5\times(-6)$.	(*d*) $\frac{1}{2}\times(-\frac{1}{3})$.	(*g*) $(-\frac{1}{2})^3$.
(*b*) $(-5)\times(-5)$.	(*e*) $\frac{1}{2}\times\left(\frac{-1}{3}\right)$.	(*h*) $(-3)^2+2$.
(*c*) $(-5)^2$.	(*f*) $\frac{1}{2}\times(-2)\times(-\frac{1}{3})$.	(*i*) $(-3)^2-10$.

8. State the value of each of the following expressions when $p = 1$.

(a) $p+1$.	(d) $2-p$.	(g) $p(-p)$.	(j) p^5.
(b) $p-1$.	(e) $2-2\,p$.	(h) $\dfrac{p-2}{p+2}$.	(k) $(-p)^6$.
(c) $p-2$.	(f) p^2-1.	(i) $\sqrt{4\,p^2}$.	(l) p^3-p^2+p-1.

9. State the value of each of the parts of Ex. 8 when $p=-2$.

10. State the value of each of the following expressions when $p=1$ and $q=-2$.

(a) $p+q$.	(d) $2\,p+q$.	(g) $pq+1$.	(j) $\dfrac{2\,p-q}{4\,p}$.
(b) $p-q$.	(e) $3\,p-2\,q$.	(h) $2\,p^2q^2$.	(k) q^3-p^3.
(c) $q-p$.	(f) p^2+q^2.	(i) $2\,pq^2$.	(l) $2\,p+\sqrt[3]{4\,q}$.

11. Write $xx^2x^3+2\,y^2y^3$ in its simplest form.

12. Find the value of each of the following expressions when $x=1$ and $y=2$.

(a) $6\,x^2y-9\,xy+y^3$.

(b) $\frac{1}{2}\,x^3-xy^2-4\,y^3$.

(c) $3\sqrt{x}-4\sqrt{2\,y}-2\,x^2+4\,y$.

(d) $\dfrac{3\,x^2-\sqrt[3]{4\,y}+\sqrt{4\,x}}{x^3+y^3}$.

13. In arithmetic the least of all numbers used is 0. Is there any such thing as the least of all numbers used in algebra? Explain.

14. If any number be subtracted from a larger one, what is the sign of the result?

15. If any number be subtracted from a smaller one, what is the sign of the result?

16. What is the number whose negative is the same as its positive?

For further exercises on this chapter, see Appendix, pp. 291–293.

CHAPTER III

ADDITION AND SUBTRACTION

28. Definitions. A *term* of an expression is one of its elementary parts; that is, a part separated from other parts by the signs $+$ or $-$. For example, the terms of the expression $3\,x+2$ are $3\,x$ and 2; the terms of $5\,ab-6$ are $5\,ab$ and 6.

Like Terms are those that contain a common factor. Thus, $3\,x$, $4\,x$, and $-5\,x$ are like terms because they contain the common factor x. Likewise, $2\,ab$, $3\,ab$, $4\,ab$, and $11\,ab$ are like terms because they contain the common factor ab.

29. Adding Like Terms. If we observe what was said in § 11 about adding like numbers, we see that like terms may always be added by merely adding their separate coefficients to obtain a new coefficient, and then multiplying that new coefficient by the common factor.

For example, suppose the like terms to be added are $2\,x$, $3\,x$, $-5\,x$, $-4\,x$, and $9\,x$. In this case, if we add the separate coefficients, we have $2+3+(-5)+(-4)+9$, which reduces (by § 17) to 5. Hence the answer is $5\,x$.

As another example, let us find the sum of the like terms $2\,r^2$, $-3\,r^2$, $2\,r^2$, $-7\,r^2$, and $4\,r^2$. The work may be arranged as follows.

Adding the separate coefficients gives

$$2+(-3)+2+(-7)+4=-2.$$

Therefore, the sum of the given terms is $-2\,r^2$. *Ans.*

Thus, we have the following rule.

RULE FOR ADDING LIKE TERMS. *Add the coefficients for the new coefficient and multiply it by the common factor.*

ORAL EXERCISES

State the sum in each of the following exercises.

1. $2x$ and $-3x$.

2. $2r$ and $-5r$.

3. $-3r$ and $7r$.

4. $-4x$, $-5x$, and $-6x$.

5. $-3y^2$, $2y^2$, and $-7y^2$.

6. $6A$, $7A$, and $-A$.

7. $-3x$, $-7x$, and $10x$.

8. $4ab$, $7ab$, and $-2ab$.

9. rs^2, $3rs^2$, and $-10rs^2$.

10. $-S$, $-10S$, $-4S$, and $21S$.

11. $6r^2x$, $4r^2x$, $-7r^2x$, and $-3r^2x$.

12. aby, $-aby$, $4aby$, and $-4aby$.

13. $$\begin{array}{r} -4\,ab \\ 7\,ab \\ -8\,ab \\ 2\,ab \\ \hline \end{array}$$

14. $$\begin{array}{r} -rs \\ 10\,rs \\ -3\,rs \\ 2\,rs \\ \hline \end{array}$$

15. $$\begin{array}{r} 3\,aby \\ -2\,aby \\ -7\,aby \\ 4\,aby \\ \hline \end{array}$$

16. $$\begin{array}{r} x^3r \\ -4\,x^3r \\ -7\,x^3r \\ -3\,x^3r \\ \hline \end{array}$$

WRITTEN EXERCISES

Simplify the following expressions by uniting terms.

1. $6x-2x+4x-7x+10x$.

[Hint. This is the same as $6x+(-2x)+4x+(-7x)+10x$.]

2. $-2b+3b+7b-b-7b$.

3. $6ar-7ar+15ar-2ar-ar$.

4. $11x^4y-2x^4y+8x^4y-3x^4y+2x^4y$.

5. $10xz+11xz-4xz+2xz+3xz-9xz$.

6. $8A+10A-3A-6A+4A-12A$.

7. $-a^2+\frac{3}{4}a^2-\frac{1}{2}a^2$.

8. $\frac{1}{2}xy+\frac{1}{8}xy-\frac{1}{4}xy$.

9. $\frac{2}{5}r^3-\frac{4}{15}r^3-\frac{1}{10}r^3$.

10. $\frac{3}{4}c+\frac{1}{7}c-\frac{2}{21}c-\frac{1}{14}c$.

11. $5.7z-2.3z+8.3z$.

12. $6.17z+2.13z-3.04z$.

13. $-1.5x^2z+6.5x^2z-x^2z$.

14. $-5A+2.5A-1.7A$.

15. $2.3x^2-1.2x^2+.4x^2-2.1x^2$.

16. $15(x+y)-2(x+y)+6(x+y)-8(x+y)$.

30. Adding Unlike Terms. When terms are unlike, that is, when they do not all have a common factor, we can no longer express their sum in one term. In such cases we can only *indicate* the addition. Thus, the sum of $3x$ and $4y$ must be written out in the form $3x+4y$. In the same way, the sum of $5a^2by$ and $2rst$ is $5a^2by+2rst$.

WRITTEN EXERCISES

Express each of the following expressions in as few terms as possible.

1. $3x+2y-7x-3x+4y$.

SOLUTION. Adding the x terms alone gives $3x-7x-3x=-7x$.
Adding the y terms alone gives $2y+4y=6y$.
Hence the result is $-7x+6y$. *Ans.*

2. $2a-3b-4a-3b$.
3. $4x+6-2x+7+9x-15$.
4. $23r-15+9r+6-17r-2$.
5. $8x^2y-xy^2+7xy^2-x^2y-4xy^2$.
6. $a-2b+2c-2a+3b-4c$.
7. $2a+2b+2c+2d-a-3b-c-3d$.
8. $mno+2m^2n^2o^2-3mno-4m^2n^2o^2+g$.
9. $1.2a-2.4b+2.3a+1.5b$.
10. $3.26x^2-2.5y^2-1.75x^2$.

For further exercises on this topic, see Appendix, p. 294.

31. Polynomials. An expression which contains more than one term is called a ***polynomial***. For example, $3x+y$, $3x+2y-4z$, and $6x-y+2z-m$ are polynomials.

When a polynomial contains only *two* terms it is called a ***binomial***. For example, $3x+y$, $x+y$, $4a-b$, $6p+4q$, are binomials.

When a polynomial has only *three* terms it is called a ***trinomial***. For example, $3x+2y-4z$, $m+n-p$, $4g-6h+2i$, are trinomials.

32. Addition of Polynomials. Polynomials are added by uniting terms that are alike. The process is similar to the adding of denominate numbers in arithmetic.

For example, adding	6 yd. + 1 ft. + 3 in.
and	2 yd. + 1 ft. + 8 in.
gives	8 yd. + 2 ft. + 11 in. *Ans.*

$$\begin{array}{rl} \text{In the same way, adding} & 2a + 7b + 3c \\ \text{and} & \underline{6a + 3b + 2c} \\ \text{gives} & 8a + 10b + 5c. \quad \textit{Ans.} \end{array}$$

These illustrations show that the rule for adding polynomials is as follows:

RULE FOR THE ADDITION OF POLYNOMIALS. *Write like terms in the same column, find the sum of the terms in each column separately, then connect the sums thus obtained by the proper signs.*

33. Arrangement of Terms in a Polynomial. A polynomial is said to be arranged in the *descending* powers of some letter when the exponents of that letter decrease as we read the polynomial from left to right, as, for example, in $x^4+3x^3+2x^2+7x+9$.

A polynomial is said to be arranged in the *ascending* powers of some letter when the exponents of that letter increase as we read the polynomial from left to right, as, for example, in $9+7x+2x^2+3x^3+x^4$.

Before adding polynomials which contain several powers of the same letter it is best to arrange all terms according to the descending (or ascending) powers of that letter.

Thus, the expressions $7x-4x^2-9+x^3$ and $-5x^2-4x+3x^3+7$ may be added as follows. Note that the terms are first arranged according to the descending powers of x.

$$\begin{array}{r} x^3-4x^2+7x-9 \\ \underline{3x^3-5x^2-4x+7} \\ 4x^3-9x^2+3x-2. \quad \textit{Ans.} \end{array}$$

34. The Checking of Addition. To check, or test, the work of addition we use special values of the letters and see if the result is correct for such values. This is illustrated in the following example.

EXAMPLE. Add the expressions

$$3a+4b+2c,\ 5a+3b-2c, \text{ and } 7a-9b-5c$$

and check your answer when $a=1$, $b=2$, and $c=3$.

SOLUTION. Adding, as in § 32, we have

$$\begin{array}{r} 3a+4b+2c \\ 5a+3b-2c \\ \underline{7a-9b-5c} \\ 15a-2b-5c. \end{array} \quad Ans.$$

CHECK. When $a=1$, $b=2$, $c=3$ the value of $3a+4b+2c$ is $3\times1+4\times2+2\times3$, which reduces to 17. Likewise, $5a+3b-2c$ then becomes equal to 5, while $7a-9b-5c$ becomes equal to -26 The sum of the three is, therefore, $17+5-26$, which reduces to -4. But, the answer obtained above, namely $15a-2b-5c$, when likewise considered for $a=1$, $b=2$, $c=3$, becomes $15\times1-2\times2-5\times3$, or $15-4-15$, and this also reduces to -4. Since the two results are the same (that is, each is -4) the work checks.

WRITTEN EXERCISES

Add the following.

1. $\begin{array}{r} 2a-3b \\ \underline{3a+8b} \end{array}$ (Check when $a=1$, $b=1$.)

2. $\begin{array}{r} -4x+3y \\ \underline{7x-8y} \end{array}$ (Check when $x=2$, $y=1$.)

3. $\begin{array}{r} 3r+2s+6t \\ 8r-5s-9t \end{array}$ (Check when $r=1$, $s=2$, $t=3$.)

4.
$$\begin{array}{r} -5\,H+\,I+10\,K \\ 7\,H-9\,I+8\,K \\ \underline{10\,H+4\,I-8\,K} \end{array}$$
(Check when $H=1$, $I=3$, $K=2$.)

5.
$$\begin{array}{r} 3\,m+7\,n+8\,p \\ -5\,m+4\,n-10\,p \\ \underline{9\,m-11\,n+5\,p} \end{array}$$
(Check when $m=2$, $n=2$, $p=2$.)

6.
$$\begin{array}{r} 6\,R+3\,S-4\,P \\ 2\,R+8\,S+2\,P \\ \underline{-3\,R-4\,S-6\,P} \end{array}$$
(Check when $R=1$, $S=1$, $P=3$.)

7.
$$\begin{array}{r} 4\,A+4\,B+4\,C \\ 2\,A-\,B-5\,C \\ \underline{A+2\,B+6\,C} \end{array}$$
(Check when $A=3$, $B=2$, $C=1$.)

8.
$$\begin{array}{l} a^2+2\,ab+b^2 \\ \underline{a^2-2\,ab+b^2} \end{array}$$
(Check when $a=2$, $b=2$.)

9.
$$\begin{array}{r} 2\,a^2-3\,ab+7\,b^2 \\ 4\,a^2+5\,ab-6\,b^2 \\ \underline{7\,a^2-3\,ab+2\,b^2} \end{array}$$

10.
$$\begin{array}{r} 3\,a+4\,b-2\,c-3\,d \\ 4\,a-3\,b+7\,c+2\,d \\ 6\,a+2\,b-\,c-\,d \\ 10\,a-5\,b+6\,c+\,d \end{array}$$

11. $3\,r+4\,s+2\,t$ and $6\,r-3\,s+5\,t$.

12. $6\,q+5\,r+z$, $q-5\,r-z$, and $-7\,q+2\,r+2\,z$.

13. $8\,x^2+4\,x+7$, $2\,x^2-3\,x-5$, and x^2-x-1.

14. $a^4+5\,a^2+3\,a$, $2\,a^3+6\,a^2-a$, and $a^3+2\,a+1$.

[HINT. The answer should contain five terms.]

15. $1-2\,r+3\,r^2$, $2+3\,r+4\,r^2$, and $-1+5\,r-3\,r^2$.

16. $x+4\,x^3-2-5\,x^2$, $6-2\,x+3\,x^2-x^3$, and $2+x^2+3x^3-x$.

[HINT. See § 33.]

17. $6a^4-a^3+2a^2+4$, $-4-2a^3+a^2-6a$, and $5a^4-4a^2+2$.

18. $a^2b^2+2ab^3+4b^2$, $5a^2b^2-2ab^3-2b^2$, and $-3a^2b^2+2ab^3+3b^2$.

19. $2H+P$, $3H-W$, $3W-5X$, and $3H+2P+5W$.

20. $2c-7d+6n$, $11m-3c-5n$, $7n-2d-8c$, $8d-3m+10c$, $4d-3n-8m$, $m-6n$, and $2m-3d$.

21. $4x^3-2x^2-7x+1$, x^3+3x^2+5x-6, $4x^2-8x^3+2-6x$, $2x^3-2x^2+8x+4$, and $2x^3-3x^2-2x+1$.

22. $a^5+5a^4b+5ab^4+b^5$, $a^4b-2a^5+a^3b^2-2b^5$,
$a^3b^2-3a^2b^3-4a^4b-a^5$,
and $2a^5+a^4b-2a^3b^2+2a^2b^3-3ab^4+b^5$.

35. Subtraction of Monomials. A ***monomial*** is an expression which contains but *one* term. Thus, x, $8y$, $2m^2$, ab, cd^3q, $lm^2n^3r^4$, $16gh^2ijk^3$ are each monomials. Compare this definition with the definitions of binomial, trinomial, etc., in § 31.

To subtract one monomial from another we simply change the sign of the subtrahend and then proceed as in addition. (See § 21.)

Thus,
$$4a-2a=4a+(-2a)=2a. \quad \textit{Ans.}$$
$$8x^2y-(-3x^2y)=8x^2y+(3x^2y)=11x^2y. \quad \textit{Ans.}$$
$$-5x^5-3x^5=-5x^5+(-3x^5)=-8x^5. \quad \textit{Ans.}$$
$$-7a^2b-(-4a^2b)=-7a^2b+4a^2b=-3a^2b. \quad \textit{Ans.}$$

In the subtraction of *unlike* terms we can only indicate the result. (Compare § 30.)

Thus, to subtract y from x we must write simply $x-y$. To subtract $3b$ from $7a$ we write $7a-3b$.

In all cases of the subtraction of monomials we therefore have the following rule.

Rule for Subtracting One Monomial from Another.

Change the sign of the subtrahend (or part subtracted) and add the result to the minuend.

ORAL EXERCISES

In solving an example in subtraction, the change of the sign of the subtrahend should be made *mentally.* The written sign should not be changed.

Subtract:

1. $\begin{array}{r} 10\,a \\ \underline{-3\,a} \end{array}$
2. $\begin{array}{r} -4\,b \\ \underline{2\,b} \end{array}$
3. $\begin{array}{r} 7\,x \\ \underline{8\,x} \end{array}$
4. $\begin{array}{r} 15\,ab \\ \underline{-16\,ab} \end{array}$
5. $\begin{array}{r} -25\,abc \\ \underline{-5\,abc} \end{array}$
6. $\begin{array}{r} 17\,a^2b \\ \underline{-4\,a^2b} \end{array}$
7. $\begin{array}{r} -19\,n^5 \\ \underline{-19\,n^5} \end{array}$
8. $\begin{array}{r} 81\,qr \\ \underline{-81\,qr} \end{array}$
9. $\begin{array}{r} 55\,a^3c \\ \underline{44\,a^3c} \end{array}$
10. $\begin{array}{r} -37\,x^3y^3 \\ \underline{-15\,x^3y^3} \end{array}$
11. $\begin{array}{r} -21\,ab^2 \\ \underline{2\,ab^2} \end{array}$
12. $\begin{array}{r} 17\,x^5 \\ \underline{-x^5} \end{array}$

13. Subtract $-7\,yz$ from $-4\,yz$.
14. Subtract $-yk$ from $-7\,yk$.
15. Subtract ab from $-ab$.
16. Subtract $21\,r^2$ from $12\,r^2$.
17. From $26\,xyz$ take $-13\,xyz$.
18. From $-11\,b$ take $4\,b$.
19. From $-9\,x^3y$ take $-7\,x^3y$.
20. From $18a^3$ take $11\,a^3$.
21. $3\,a-(-5\,a)=$?
22. $-6\,xy-(-4\,xy)=$?
23. $8\,r-(-2\,r)=$?
24. Take $-4\,z$ from $-6\,z$.

State the answer in each of the following four exercises. Note that these deal with *unlike* terms. (See § 35.)

25. Subtract $3\,b$ from $10\,a$.
26. Subtract $13\,r^2$ from $-14\,st$.
27. Subtract $-7\,xy^2$ from y^2z^2.
28. Subtract $90\,pqrs$ from $abcd$.

For further exercises on this topic, see Appendix, p. 295.

36. Subtraction of Polynomials. To subtract one polynomial from another, write like terms in the same column and subtract in each column separately. For example, in subtracting $3x-8y$ from $2x+4y$ the work is as follows:

$$\begin{array}{r} 2x+4y \\ 3x-8y \\ \hline -x+12y. \end{array} \quad \textit{Ans.}$$

To check your answer, show that the sum of the remainder (or answer) and the subtrahend equals the minuend. Thus, in the example just worked, if we add $-x+12y$ to $3x-8y$, we should get $2x+4y$. Do we?

If the subtrahend and the minuend contain no like terms, we can only *indicate* the subtraction. Thus, in subtracting $6x+y$ from $2m+n$ the result is $2m+n-6x-y$. In all cases (whether the subtrahend and minuend contain like terms or not) the result can always be obtained by merely changing all signs in the subtrahend and adding the result to the minuend. Hence, we have the following rule.

RULE FOR SUBTRACTING ONE POLYNOMIAL FROM ANOTHER. *Change all signs in the subtrahend and add the result to the minuend.*

NOTE. Polynomials which contain different powers of some letter should first be arranged according to the descending or ascending powers of that letter. (Compare § 33.)

WRITTEN EXERCISES

Perform each of the following subtractions, and check your answer in each.

1. $\begin{array}{r} 9a+7b \\ 2a+3b \end{array}$	2. $\begin{array}{r} 4a+10b \\ 5a+4b \end{array}$	3. $\begin{array}{r} 10x+2y \\ 6x+4y \end{array}$	4. $\begin{array}{r} 4m+4n \\ 2m+5n \end{array}$

5. $\begin{array}{r} 11\,m-2\,n \\ \underline{9\,m-\,n} \end{array}$	**6.** $\begin{array}{r} 9\,a-16\,b \\ \underline{3\,a+22\,b} \end{array}$	**7.** $\begin{array}{r} 10\,a-2\,c \\ \underline{-16\,a-14\,c} \end{array}$	**8.** $\begin{array}{r} 2\,r^2+8\,ab^3 \\ \underline{5\,r^2-7\,ab^3} \end{array}$

9. From $8\,p+3\,z$ subtract $10\,p+z$.

10. From $-10\,m+n$ subtract $-5\,m-3\,n$.

11. From $3\,ax-5\,by$ subtract $-4\,ax+6\,by$.

12. From $7\,abc+10\,mx$ subtract $20\,abc-3\,mx$.

13. From $4\,m-3\,n+2\,p$
take $\underline{2\,m-5\,n-\,p}$

14. From $3\,a-4\,b+7\,c$
take $\underline{6\,a-5\,b+9\,c}$

15. From $a-b+c$
take $\underline{2\,a+b-c}$

16. From x^2-xy+y^2
take $\underline{x^2+xy+y^2}$

17. From $a^2+2\,ab+b^2$
take $\underline{a^2-2\,ab+b^2}$

18. From $8\,a^2b-5\,ac^2+9\,a^2c$
take $\underline{3\,a^2b+2\,ac^2-9\,a^2c}$

19. From $10\,m-6\,n-3\,p$ subtract $2\,n+3\,p-4\,m$.

[Hint. First rearrange the subtrahend in the form $-4\,m+2\,n+3\,p$.]

20. From $6\,ac+10\,bd-8\,s$ subtract $4\,s-10\,bd+5\,ac$.

21. Subtract $-5\,x+8\,x^3+7+2\,x^2$ from $-2+6\,x-6\,x^2+4\,x^3$.

[Hint. See Note in § 36.]

22. Take $7\,xy^2+9\,x^2y-7\,x^2y^2$ from $4\,x^2y-6\,x^2y^2-12\,xy^2$.

23. From $2\,ax-by-xy$ take $2\,by-2\,ax-3\,xy$.

24. From $2\,a+3\,c+d$ subtract $a-b+c$.

Solution. Here the subtrahend and the minuend contain some *unlike* terms, and the work would therefore look as follows:

$$\begin{array}{l} 2\,a+3\,c+d \\ a+c-b \\ \hline a+2\,c+d+b. \end{array}$$

Ans. (See Rule in § 36.)

25. From $a-3\,b+c$ subtract $a+c-d$.

26. From $2\,x+2\,xy$ take $xy+x-y$.

27. From $2\,a-2\,d$ subtract $a-b+c-d$.

28. From $1+2\,a+3\,a^2+6\,a^3$ take $3\,a+4-2\,a^2$.

29. From $1-a^3$ take $1-a+a^2-a^3$.

30. From $4\,ab+c$ subtract $5\,m-2\,n+3\,q$.

[HINT. Since the subtrahend here contains *no* terms like those in the minuend, no special arrangement of the work is necessary. Thus the answer is $4\,ab+c-5\,m+2\,n-3\,q$.]

31. From $xy-5\,yz^2$ subtract $xy^2+y^2z-8\,x^2y^2-7\,yz^2$.

32. From $1-2\,x^3+x^2$ subtract $x^4+4\,x^2-5-3\,x$.

33. From $3\,x-y+z$ subtract the sum of $x-4\,y+z$ and $x+4\,y+z$.

34. From the sum of $2\,s+8\,t-4\,w$ and $3\,s-6\,t+2\,w$ subtract the sum of $8\,s+9\,t+6\,w$ and $4\,s-7\,t-4\,w$.

35. From $\frac{2}{3}\,x^3-\frac{2}{5}\,x^2+x-5$ subtract $\frac{1}{2}\,x^3-\frac{4}{5}\,x^2-2\,x+1$.

For further exercises on this topic, see Appendix, p. 295.

37. Further Study of Equations. In §§ 6, 7, 8 we saw that numbers may be added to, or subtracted from, each side of an equation, and it was in this way that we *solved* the equation; that is, found the value of the unknown letter. Any equation was shown to be like a pair of balance scales, the balance being undisturbed so long as both sides of the scales were changed *exactly* alike, and the usual ways of doing this were stated in the axioms of § 9. However, the only equations we learned how to solve were those having *positive* answers. The same axioms apply also to solving equations whose answers are negative, as we shall now show by means of two examples.

EXAMPLE 1. Solve the equation $7x+27=3x-9$.

SOLUTION. $7x+27=3x-9$. (Given)

Subtracting $3x$ from both sides, we have

$4x+27=-9$. (Axiom II, § 9)

Subtracting 27 from both sides of the last equation, we find

$4x=-36$. (Axiom II)

Dividing both sides by 4, we have

$x=-9$. *Ans.* (Axiom IV)

CHECK. With $x=-9$ the left side of the given equation becomes $7(-9)+27$, or $-63+27=-36$. At the same time its right side becomes $3(-9)-9$, or $-27-9=-36$. Since both sides thus become the same (namely, -36), the answer must be correct; that is, -9 is the desired value of x.

EXAMPLE 2. Solve the equation $3x-9=8x+20$.

SOLUTION. $3x-9=8x+20$. (Given)

Subtracting $8x$ from both sides, we have

$-5x-9=20$. (Axiom II)

Adding 9 to both sides, we find

$-5x=29$. (Axiom I)

Dividing both sides by -5, we have

$x=-5\frac{4}{5}$. *Ans.* (Axiom IV)

EXERCISES

Solve the following and check your answer for each.

1. $x+8=10$.
2. $x+8=7$.
3. $4-x=7$.
4. $6-x=3$.
5. $-4+x=8$.
6. $-4+x=-8$.
7. $4-x=-3$.
8. $-x+5=24$.
9. $x+1=-1$.
10. $8x+2x=-12$.
11. $4x-9x=25$.
12. $10-2x=4$.
13. $2x-18=4x$.
14. $4x=6x-22$.
15. $3x-10=20$.
16. $2x+16=8x+4$.
17. $13x-4=18x+14$.

18. $7K+36=-2K-90$.

19. $-50r+2r=100+77r$.

20. $8y-16=3y+30$.

21. $6z+15-4z=21+3z-8$.

22. $12k-9+32=24k-13+k$.

23. $7b+13=43-2b$.

24. $-3a+6=a+18$.

25. $-10r+15=-25$.

26. $6\frac{1}{2}x-8+2\frac{1}{2}x=-17$.

27. $2.3r-4.6=1.2r+3.2$.

28. The sum of two numbers is 12 and one of them is twice the other. Find the numbers.

[HINT. Let x = the smaller number. Then $12-x$ = the larger number.]

29. The sum of two numbers is 12 and their difference is 14. Find the numbers.

30. If a certain number be subtracted from 12 the remainder is 19. What is the number?

31. If 8 be subtracted from four times a certain number, the result is 16 more than twice the number. Find the number.

32. The sum of three numbers is 21. The second is 6 more than the first, and the third is 2 less than the first. Find the numbers.

[HINT. Let x = the first number.]

33. The sum of three numbers is 21. The second is 15 more than the first, and the third is 19 less than the second. Find the numbers.

34. The length of a certain rectangle is 4 feet more than twice the width. The whole distance around (called perimeter) is 56 feet. What is the length and what the width?

35. A rectangle whose perimeter (see Ex. 34) is 98 feet is 18 feet longer than wide. Find its dimensions (length and breadth)

36. Find two *consecutive* numbers whose sum is 17.

[HINT. Two numbers are said to be *consecutive* when the one is 1 greater than the other. Thus, 2, 3 are consecutive; so also are 10, 11; so also are -3, -2. More generally, *several* numbers are consecutive when each is 1 greater than the one before it. Thus, 2, 3, 4, 5, 6 are consecutive; so also are -2, -1, 0, 1, 2, 3, 4; so also are $\frac{1}{3}$, $\frac{4}{3}$, $\frac{7}{3}$, $\frac{10}{3}$.]

37. Find three consecutive numbers such that their sum is equal to the last number increased by 17. (See Hint to Ex. 36.)

38. Find four consecutive numbers such that the sum of all four of them is the same as the sum of the first and last.

For further exercises on this topic, see Appendix, pp. 293–295.

CHAPTER IV

PARENTHESES

38. Definition. Parentheses ( ) are used to show that the terms included within them are to be regarded as one number. Thus, $6+(3-2)$ means that we are to add $3-2$ or 1, to 6. That is, $6+(3-2)=6+1=7$. *Ans.*

Similarly, $6-(3-2)$ means that we are to subtract $3-2$, or 1, from 6. That is, $6-(3-2)=6-1=5$. *Ans.*

Other examples, which should be carefully examined, follow.

$$(2+3)+(2-1)=5+1=6. \quad \textit{Ans.}$$
$$(2-3)+(5-7)=-1+(-2)=-3. \quad \textit{Ans.}$$
$$4-(5-7)=4-(-2)=4+2=6. \quad \textit{Ans.}$$
$$(a+2)+(a-3)=a+2+a-3=2\,a-1. \quad \textit{Ans.}$$

39. To Multiply a Quantity Inclosed in Parentheses. When a number is placed directly before or after an expression inclosed in parentheses, with no sign between them, multiplication is indicated.

Thus, $2(4-2)$ means $2\times(4-2)$; that is, 2×2, or 4.

Similarly, $3(a+5)$ means that the sum of a and 5 is multiplied by 3. Again, $(a+2)(a-5)$ means that the sum of a and 2 is multiplied by the difference between a and 5.

ORAL EXERCISES

1. Give the value of each of the following expressions.

(*a*) $4+(3-1)$.

(*b*) $4-(3-1)$.

(*c*) $(2+3)+(5-6)$.

(*d*) $(2+3)-(1-2)+(3+4)$.

(*e*) $8(4-3)$.

(*f*) $(2-3)(4-5)$.

(*g*) $7(6-1)(4-5)$.

(*h*) $2(3-1)+(6-7)$.

2. Read each of the following expressions.

(*a*) $5(a+2)$.
(*b*) $6(a-b)$.
(*c*) $(a+3)(a+2)$.
(*d*) $4(a^2-2)(a^2-1)$.
(*e*) $(a-1)(a^2-2)(a^3-3)$.
(*f*) $(a-b)^2(a-b+2)^2$.

WRITTEN EXERCISES

1. Find the value of each of the parts of Ex. 2 in the last set of exercises when $a=1$ and $b=2$.

2. Write out the algebraic expression for each of the following phrases.

(*a*) Four times the sum of r and s.
(*b*) Two times the difference between a and b.
(*c*) The sum of a and b multiplied by their difference.
(*d*) The square of the sum of a and b.
(*e*) The square of the difference between a and b.
(*f*) The sum of x square and y square multiplied by the sum of a and b.
(*g*) The sum of x square, y square, and z square multiplied by the difference between r cube and s cube.
(*h*) Seven times the difference between x square and y square multiplied by the sum of f, g, and h.
(*i*) The sum of the squares of a, b, and c multiplied by the sum of the cubes of x, y, z, and w.
(*j*) The cube of the difference between a and b.
(*k*) The cube of the sum of a, b, and c.

40. Removing Parentheses. Such an expression as $10+(6+2)$, in which the sign before the parentheses is $+$, may be simplified in either of two ways:

(1) By getting the value of the part in parentheses and adding it to 10. Thus,

$$10+(6+2)=10+8=18.$$

(2) By removing the parentheses and simplifying the result thus obtained. Thus,

$$10+(6+2)=10+6+2=18.$$

Other illustrations, which should be carefully examined, follow.

$2+(4-1)=2+3=5,$
or, $2+(4-1)=2+4-1=6-1=5.$

$3+(5-7)=3+(-2)=1,$
or, $3+(5-7)=3+5-7=8-7=1.$

ORAL EXERCISES

Work mentally each of the following exercises, first with parentheses and then without.

1. $10+(7+3)$.
2. $8+(9-6)$.
3. $(6+2)+4$.
4. $(10-6)+2$.
5. $1+(2-3)$.
6. $(2+3)+(3-2)$.
7. $(3-1)+(4-3)$.
8. $-4+(3-1)$.
9. $(1-2)+(1-3)$.
10. $2+(4-3+1)$.
11. $(3-4-1)-1$.
12. $-1+3-2+(6-7)$.

In each of the examples thus far considered the parentheses have been preceded by the sign $+$. Suppose now a case where the parentheses are preceded by a $-$ sign. For example, consider the expression $10-(6+2)$. Here again we may proceed in either of two ways:

(1) By getting the value of the part in parentheses and subtracting it from 10. Thus,

$$10-(6+2)=10-8=2.$$

(2) By simply removing the parentheses, *provided, however, that we first change the sign of each term in the parentheses.* (See § 36.) Thus,

$$10-(6+2)=10-6-2=4-2=2.$$

Other illustrations follow:

$4-(3-2)=4-1=3,$
or, $4-(3-2)=4-3+2=1+2=3.$

$2-(3-4)=2-(-1)=2+1=3,$
or, $2-(3-4)=2-3+4=-1+4=3.$

ORAL EXERCISES

Work mentally each of the following, first with parentheses, then without.

1. $10-(7+3)$.
2. $8-(9-6)$.
3. $7-(1+2-4)$.
4. $(2+3)-(4-5)$.
5. $-4-(3-1)$.
6. $1+(2-1)-(3-1)$.
7. $-(3-2)-(2-3)$.
8. $-(2+3-1)+(2-1)$.
9. $2-4-(7-5)$.
10. $3+(2-4+1)-(3-7)$.

We may now state what we have just seen in the following rule.

RULE FOR REMOVING PARENTHESES. *A parenthesis preceded by the + sign (either expressed or understood) may always be removed.*

A parenthesis preceded by the − sign may be removed provided the sign of each term in the parenthesis be first changed.

EXERCISES

In each of the following exercises, remove the parentheses and reduce to the simplest form.

1. $a+(b-c)-(2a+3b)$.

SOLUTION. Following the Rule of § 40, the result of removing the parentheses is $a+b-c-2a-3b$. Combining like terms, this becomes $-a-2b-c$. *Ans.*

2. $2a-(b-c)+(3c-d)$.
3. $(x^2-y^2)-(x+y-z)$.
4. $-(m^2-n^2)+(m-n^2+pq)$.
5. $a+2b+3c-(a+b+c)-(2a+3b-2c)$.
6. $a^2b+b^2c+ac^2-(2ab^2-3a^2c)+(4a^2b-5a^2c^2-6a^2b^2)$.

41. Bracket. Brace. Vinculum. When a group of terms is included within another group, it becomes necessary to use some other form than parentheses. The *bracket* [ ], the *brace* { }, and the vinculum $\overline{\quad}$ are used for such purposes. For example, the expression

$$a+[r-\{6+(b+c-d)\}]$$

means that the group $(b+c-d)$ is first to be added to 6, then the result (considered as a new group) is to be subtracted from r, then this result (considered as a new group) is finally to be added to a.

42. Removing Group Signs. When an expression contains various group signs, such as the parenthesis, the bracket, brace, etc., they may all be removed *in succession*, beginning either with the outermost or innermost, preferably the latter.

Example. Simplify $8a-[3b+4a+(-a+2b)]$.

Solution. The rule of § 40 gives

$$8a-[3b+4a+(-a+2b)]$$
$$=8a-[3b+4a-a+2b],$$

which, combining terms,

$$=8a-[3b+3a+2b],$$

which, by the rule of § 40,

$$=8a-3b-3a-2b,$$

or, by combining terms,

$$=5a-5b. \quad \textit{Ans.}$$

WRITTEN EXERCISES

Simplify each of the following expressions by removing all the signs of grouping.

1. $a-(2a+4a)-(5a+10)$.
2. $6a+(5a-[2a+1])$.
3. $10r-(4r-\{-3r-2\})$.
4. $(2r-c)-(5r-2c)$.

5. $x-\{x-(x-3\,x)\}$.

6. $8\,a-(-3\,a+4)+(-2\,a+10)$.

7. $6\,r-\{10-(2\,r+6)-r\}$.

8. $x-(10\,x-\{2\,x+4\}-6)$.

9. $20\,z-\{(2\,z+7\,r)-(3\,z+5\,r)\}$.

10. $2(8-10\,c)-[(-3+10\,c)+(2-8\,c)]$.

11. $8\,a-\{4\,a+[6\,a-(2\,a+17)]\}$.

12. $\{x-(x+[x-1]+4)+2\}$.

Find the value of each of the following expressions when $a=4$, $b=3$, $c=2$, and $d=5$. Recall the directions stated in § 15 for order of operations.

13. $a-2(c+d)$.

14. $abcd+a^2-c^2$.

15. $10\,c^2-(3\,c+2\,d)$.

16. $c+\dfrac{b+d}{2}$.

17. $3(a+b+c-d)-6\,a$.

18. $\sqrt{a+c+d-7}-\sqrt{b+c+d-1}$.

19. $(a+b)(a-b)$.

20. $(a+b)+8(d-c)$.

21. $\dfrac{a^2+b^2+c^2}{a-ab+bc}$.

22. $\dfrac{2\,a+3\,b}{c-d+1}$.

Solve each of the following equations. First remove parentheses in each.

23. $3\,s-(s-10)=40$.

24. $(3\,r-2)+(7\,r-6)=10+(2\,r+4)$.

25. $(4\,x-5)-(2\,x+7)=18-(x-1)$.

26. $2\,a-(4\,a+7)=(-a+2)-(2+5\,a)$

27. $(m-6)+(m+6)=m+3$.

28. $[2\,x-(2-x)]=3-(x-1)$.

EXERCISES — REVIEW OF CHAPTER IV

I. Oral Exercises

Read each of the following expressions.

1. $5(x-y+z)$.

2. $(a+b)(a-b)$.

3. $(a^2+b^2)(a^2-b^2)$.

4. $(a+b)^2(a-b)^2$.

5. $6(x-y)(p-q+r)$.

6. $(m-n)(m^2-n^2)(m^3-n^3)$.

7. $(\sqrt{x}-1)(\sqrt{y+1}+1)$.

8. $\sqrt[3]{(x-1)(y^2+4)}$.

9. $(x^5-x^4)\sqrt{5(y^2+3)}$.

10. $x^3-(3-2x^2)^2$.

11. State the value of the expression in Ex. 7 when $x=1$, $y=3$.

12. Do a^2+b^2 and $(a+b)^2$ mean the same thing? Are they equal? Explain.

13. Are $a+b-c$ and $-(c-a-b)$ equal to each other? Explain.

14. From among the following expressions pick out those which are equal to $a+b-c$.

(*a*) $(a+b)-c$.
(*b*) $(a-c)+b$.
(*c*) $-c-(b-a)$.
(*d*) $a+(b-c)$.
(*e*) $b-(a+c)$.
(*f*) $-c+(a+b)$.
(*g*) $a-(c-b)$.
(*h*) $-c-(a-b)$.
(*i*) $(b-c)-(-a)$.

15. State such ways as you can for writing $x-y+z$, using parentheses.

II. Written Exercises

When $a=1$, $b=2$, $c=3$, $d=4$, and $e=5$ find the value of each of the following expressions.

16. $a-(e+b)-(c+d)-(e-d+b+c)$.

17. $3ab^2-2be^3-(d^ce^a-dc^2)+8be^2$.

18. $\sqrt{2edb+4e}-\sqrt[3]{9a^cc^4}-2c^2d-(abe-abcde)$.

19. Solve the equation $2x-(4+x)-5x+20=4x+(4-5x)$.

20. Solve the equation

$$10x-3-(4-2x)+(3x-4x+5-2x)=2-3x+4x-(2x+x)+7.$$

Remove parentheses and collect similar terms in the following.

21. $x^2-(x^2y-z^2)-z^2+(x^2y-x^2)$.

22. $x-[-\{-(-x-1)-x\}-1]-1$.

23. $-[-2x-\{-(-2x-1)-2x\}-1]-2x$.

24. $x-[x+(x-y)-\{x+(y-x)-2y\}+y]-y+x$.

25. If $A=4x^3-2x^2y+3xy^2+y^3$, $C=3x^3-x^2y+2y^3$,
$B=4x^3-x^2y-xy^2-3y^3$, $D=x^3-2xy^2+y^3$,

find the value of $A-B+C-D$.

CHAPTER V

SIMPLE EQUATIONS

43. Introduction. We have already seen in Chapter I how to solve such equations as $2x-5=x+1$. The steps are as follows:

$$2x-5=x+1. \qquad \text{(Given)}$$

Adding 5 to both sides gives

$$2x=x+6. \qquad \text{(Axiom I, § 9)}$$

Subtracting x from both sides (of the last equation) gives

$$x=6. \quad \textit{Ans.}$$

CHECK. With $x=6$ the left side of the given equation becomes $2\times 6-5$, or $12-5$, which is 7, while the right side becomes $6+1$, which also is 7.

This way of solving an equation step-by-step will always give the value of x, but in practice the work is usually much shortened in ways which we shall now explain.

44. Transposition. Let us look again at the preceding solution of the equation $2x-5=x+1$. The first step (which was to add 5 to both sides) practically amounts to getting rid of the -5 on the *left* side by putting $+5$ on the *right* side. In fact, when we do this we get

$$2x=x+1+5,$$

which reduces to

$$2x=x+6.$$

Likewise, the second step (which was to subtract x from both sides of the equation $2x=x+6$) practically amounts to

getting rid of the x on the right side by placing $-x$ on the *left* side. In fact, this gives

$$-x+2x=6,$$

which reduces to

$$x=6.$$

Whenever we get rid of a term on either side of an equation in this way we are said to ***transpose*** it. Much time can be saved in making use of this idea, since we can often transpose a number of different terms *all at once*, thus condensing the separate steps which would be necessary if we followed the long way of Chapter I. All we have to remember is that *every time a term is transposed its sign must be changed.* The following examples will further illustrate this idea and its usefulness.

EXAMPLE 1. Solve the equation

$$4x-2=10+2x.$$

SOLUTION. Transposing the term -2 to the right side and the $2x$ to the left side, we obtain the equation

$$4x-2x=10+2,$$

which reduces to

$$2x=12.$$

Hence we have

$$x=6. \quad \textit{Ans.} \qquad \text{(Axiom IV, § 9)}$$

EXAMPLE 2. Solve the equation

$$2x-5-(6-4x)=x-(7+3x).$$

SOLUTION. Removing the parentheses (see § 40) we find

$$2x-5-6+4x=x-7-3x.$$

Transposing all the terms containing x to the left side, and all the others to the right side, we have

$$-x+3x+2x+4x=-7+5+6.$$

Combining terms gives

$$8x=4.$$

Dividing both sides by 8 gives

$$x=\tfrac{4}{8}=\tfrac{1}{2}. \quad \textit{Ans.}$$

From what we have just seen we may state the following general principle.

PRINCIPLE. *Any term may be transposed from one side of an equation to the other, provided its sign is changed.*

HISTORICAL NOTE. Our word *algebra* comes from the Arabic word *al-gebr*, meaning "to restore." The Arabs avoided negative numbers as far as possible and the first thing they would do with such an equation as $x+2=2x-1$ was to "restore" it to a form that did not contain the -1; that is, they would write the equation over in the form $x+3=2x$. This is what we now call *transposing* the -1.

ORAL EXERCISES

1. Explain the transposition in each of the following cases:

(*a*) $3x-1=6.$	(*b*) $4x+2=12.$	(*c*) $2x=13-3x.$
$3x=6+1.$	$4x=12-2.$	$2x+3x=13.$

2. What transposition would you suggest for each of the following in order to solve it?

(*a*) $3x-5=2x+10.$ (*d*) $6=3x-5x+2.$
(*b*) $7x-4x-3=2x-2.$ (*e*) $10x-3-4x=5-x.$
(*c*) $9x=5+2x+2.$ (*f*) $2x-5+6x=0.$

45. Cancelling Terms in an Equation. Consider the equation

$$2x+7=30+7.$$

Here 7 occurs on *both* sides. Hence, if we subtract it from each side (by Axiom II, § 9) it disappears entirely from the equation; that is, the equation becomes simply $2x=30$. In such a case the 7 is said to ***cancel*** from both sides.

The way in which this idea is used in practice is illustrated in the following examples, where the cancelled terms have the mark / drawn through them.

Example 1. Solve the equation

$$3-(7-4\,x)=x+(2\,x+3).$$

Solution. Removing the parentheses (see § 40) gives

$$3-7+4\,x=x+2\,x+3.$$

Cancelling the 3 gives

$$\cancel{3}-7+4\,x=x+2\,x+\cancel{3},$$

or

$$-7+4\,x=3\,x.$$

Transposing (see § 44) gives

$$4\,x-3\,x=7,$$

or

$$x=7. \quad \textit{Ans.}$$

Example 2. Solve the equation

$$x+(3\,x+5)=5-(9-x).$$

Solution. Removing parentheses gives

$$x+3\,x+5=5-9+x.$$

Cancelling wherever possible gives

$$\cancel{x}+3\,x+\cancel{5}=\cancel{5}-9+\cancel{x},$$

or

$$3\,x=-9.$$

Dividing by 3 gives

$$x=-3. \quad \textit{Ans.}$$

ORAL EXERCISES

State where cancellation can take place in each of the following cases.

1. $5\,x+3=20+3.$
2. $5\,x-3=20-3.$
3. $5\,x+x=x-20.$
4. $4+3\,x-x=6+4-x.$
5. $5-(3-x)=-2\,x+x-3.$

   [Hint. First remove the parentheses.]
6. $2\,x-(3\,x-x-7)=3\,x+(x+3).$
7. $(3\,y-4)-[2-(y-4)]=(3\,y-4)-2.$
8. $2\,z-[z+(1-z)]=2\,z-(1+z).$

46. Changing Signs throughout an Equation. The signs of all the terms of an equation may be changed without destroying the equality. Thus, $6-4x=10-3x$ is equivalent to $-6+4x=-10+3x$. In fact, all we have to do to get this last equation is to multiply both sides of the first equation by -1. (Axiom III, § 9.)

Changing signs throughout an equation in this way is often useful, as illustrated below.

EXAMPLE. Solve the equation

$$-4x+3=-x-9.$$

SOLUTION. Changing signs throughout gives $4x-3=x+9$.
Transposition gives

$$3x=12,$$

and hence

$$x=4. \quad Ans.$$

47. Definitions. Any equation which, like those we have thus far considered, contains no higher power of the unknown letter than the first is called a ***simple equation***, or an ***equation of the first degree***, or a ***linear equation***.

Thus, $3x-4=4x-5$ is a simple equation, but $3x^2-4=4x-5$ is *not* a simple equation. Other examples of equations that are not simple are: $x^3+2x^2=x+4$; $3x^3-4=x+5$; $x^4+3x^2-x=7$.

Equations containing no higher power of the unknown letter than the *second* will be considered in later chapters.

The value of the unknown letter in a simple equation is called the *root*, or the *solution*, of the equation. To *solve* a simple equation is (as we have seen) to find its root.

The usual process of solving consists in making full use of the short methods explained in this chapter, such as transposition, cancellation of terms, etc., so as to arrive quickly at the value of the unknown letter.

WRITTEN EXERCISES

Solve each of the following equations, making use of the short methods explained in this chapter in any way you please. Check your answer.

1. $x+2=2x-1$.

[HINT. Transpose so as to have all x terms on the left and all other terms on the right. Then combine like terms and use Axiom IV.]

2. $2x+1=-x-3$.
3. $3x-3=2x+4$.
4. $\frac{1}{2}x-3=x+5$.
5. $\frac{2}{3}x+\frac{1}{9}=\frac{1}{6}x+\frac{1}{3}$.
6. $2x+(x-1)=3+(2x+3)$.

[HINT. First remove the parentheses.]

7. $x-(2x-1)=3x+1$.
8. $2x-[3-(1-x)]=-x-3$.
9. $1-\{x-(2x+1)\}=-(2x+3)$.

10. If 1 be added to twice a certain number, the result is 3 more than the number itself. What is the number?

[HINT. Work by algebra, letting x represent the unknown number.]

11. If a certain number be subtracted from three times itself, the result is 5 more than the number. What is the number?

12. Answer the last question when -5 is used instead of 5. What does the question mean in this case?

13. A certain recipe for fruit punch calls for twice as many oranges as lemons. If oranges are 35 cents a dozen and lemons are 25 cents a dozen, how many dozens of each must be used in making a punch that is to cost \$4.75?

[HINT. Let $x=$ the number of dozens of lemons used.]

14. In a certain house the parlor contains 3 more lights than the dining room, and the dining room 4 more than the kitchen, while the pantry contains one light only. If the total is 24 lights, how many are there in each of the rooms?

15. The interest on a certain sum of money at 5% amounts in one year to $1 less than it would at 6%. What is the sum?

16. The month of March contained 13 more stormy days than bright ones, and of the stormy days there were 3 more with snow than rain. How many were there of each kind?

17. In trying to find the weight of a single egg, I found that four eggs and a one ounce weight balanced against one egg and a half pound weight. What is the weight of each egg?

For further exercises on this topic, see Appendix, p. 297.

CHAPTER VI

MULTIPLICATION AND SPECIAL CASES OF FACTORING

PART I. MULTIPLICATION

48. Product of Powers of the Same Number. We have seen (§ 13) that x^2 (read x *square*, or x to the *second* power) means $x \cdot x$, while x^3 (read x *cube*, or x to the *third* power) means $x \cdot x \cdot x$.

Likewise, we explained (§ 25) that x^4 (read x to the *fourth* power) means $x \cdot x \cdot x \cdot x$.

In the same way x^5 (read x to the *fifth* power) means $x \cdot x \cdot x \cdot x \cdot x$, etc.

In each case the number above the x is called the ***exponent*** of x.

Suppose now we consider the product $x^2 \cdot x^3$. This means $(x \cdot x) \cdot (x \cdot x \cdot x)$. But this is the same as $x \cdot x \cdot x \cdot x \cdot x$, which is x^5.

Thus, we see that

$$x^2 \cdot x^3 = x^{(2+3)} = x^5.$$

Similarly, if we consider $x^3 \cdot x^4$, we may write

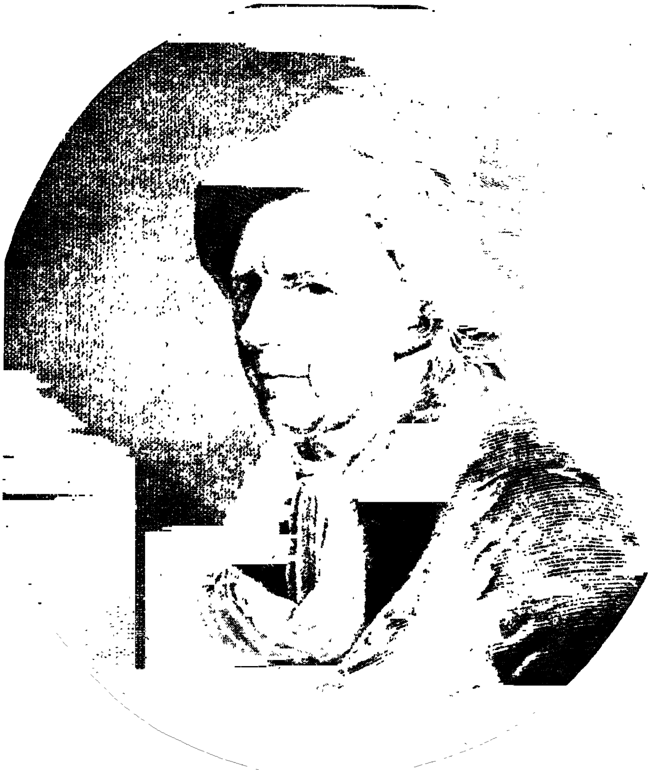
$$x^3 \cdot x^4 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^{(3+4)} = x^7.$$

We arrive in this way at the following rule.

Rule for Multiplying Powers of the same Number. *The exponent of the product of two powers of the same number is equal to the* ***sum*** *of the exponents of the factors.*

Stated as a formula, this rule becomes

Formula I. $$x^m \cdot x^n = x^{(m+n)}.$$



EULER

(*Leonhard Euler*, 1707–1783)

Revised and enlarged all the branches of mathematics known at his time. In algebra, especially famous for his studies on Infinite Series. His great work entitled *Analysis Infinitorum* contains essentially all that is to be found to-day in the textbooks on algebra and trigonometry, besides much else of a more advanced character.

ORAL EXERCISES

State the result of each of the following multiplications.

1. $a^4 \cdot a^5$ **2.** $b^3 \cdot b^8$ **3.** $c^5 \cdot c^6$ **4.** $x^3 \cdot x^6$

NOTE. In the following exercises, we omit the dot. That is, we write x^3x^6 instead of $x^3 \cdot x^6$, etc.

5. $r^{12}r^3$.

6. d^8d.

7. q^5q^{12}.

8. s^7s^3.

9. z^3z^{18}.

10. w^3w^6.

11. a^2a^m.

12. a^3a^r.

13. a^na^4.

14. a^sa^5.

15. a^sa^n.

16. $a^{2x}a^x$.

17. $a^{4x}a^{2x}$.

18. $a^{3x}a^{5x}$.

19. ax^3a^2; a^3x^3. *Ans.*

20. $a^2y^3a^2y^2$.

21. $a^3y^2a^5y^4$.

22. $(-a)^2a^7$.

[HINT. See § 26.]

23. $a^5(-a)^2$.

24. $(-a)^5(-a)^4$.

25. $(-a^2y^2) \cdot (-a^2y^4)$

26. $(m^2n^3)(-m^3n^2)$.

27. $a^{r+s} \cdot a^{r-s}$

28. $x^2x^3x^4$.

[HINT. First write $x^2x^3=x^5$.]

29. $zz^2z^3z^4$.

30. $x^{2n}x^{3n}x^{6n}$.

49. Product of Any Two Monomials.

EXAMPLE. Multiply $-3\,a^2x^2$ by $4\,a^3x^4$.

SOLUTION.

$$\begin{array}{r} -3\,a^2x^2 \\ 4\,a^3x^4 \\ \hline -12\,a^5x^6. \end{array} \quad Ans.$$

Here we have first multiplied the -3 by 4 to obtain the new coefficient, -12, then we have multiplied a^2 by a^3, and x^2 by x^4, which, by Formula I, (§ 48) gives a^5x^6. So the answer is $-12\,a^5x^6$.

Similarly, in all cases we have the following rule.

RULE FOR MULTIPLYING MONOMIALS. *To multiply one monomial by another, multiply the coefficients to obtain a new coefficient, then multiply the letters together, observing Formula I of* § 48.

ORAL EXERCISES

1. $\begin{array}{r} 10\,a^5 \\ \underline{6\,a^2} \end{array}$

2. $\begin{array}{r} -4\,m^2n^3 \\ \underline{2\,m^2n^2} \end{array}$

3. $\begin{array}{r} 3\,a^2b^2c^3 \\ \underline{-2\,a^3b^2c^2} \end{array}$

4. $\begin{array}{r} -3\,a^2b^3y^4 \\ \underline{-2\,a^3b^2y} \end{array}$

5. $\begin{array}{r} -2\,ab \\ \underline{3\,ba} \end{array}$

6. $\begin{array}{r} 4\,abc^2 \\ \underline{3\,b^2ca^3} \end{array}$

7. $\begin{array}{r} -8\,r^{s-n} \\ \underline{4\,r^{s+n}} \end{array}$

8. $\begin{array}{r} -2\,a^xb^{2y} \\ \underline{7\,a^{3x}b^{4y}} \end{array}$

9. $\begin{array}{r} -6\,a^2c^2x \\ \underline{-4\,a^3b^2n} \end{array}$

[Hint. The answer should here be in the form $24\,a^5c^2xb^2n$, as this cannot be further simplified.]

10. $\begin{array}{r} -5\,m^3d^2 \\ \underline{2\,m^{10}c^2d^3} \end{array}$

11. $\begin{array}{r} 2\,a^2m^3n^4 \\ \underline{8\,b^3n^7p^6} \end{array}$

WRITTEN EXERCISES

Carry out the following indicated multiplications.

1. $(3\,a^2bc^2)\times(-3\,abc)$. Check when $a=1$, $b=1$, $c=2$.

Solution. Multiplying as in § 49, we obtain $-9\,a^3b^2c^3$. *Ans.*

Check. When $a=1$, $b=1$, $c=2$, we see that $3\,a^2bc^2=3\times1\times1\times4=12$; likewise that $-3\,abc=-3\times1\times1\times2=-6$, and $-9\,a^3b^2c^3=-9\times1\times1\times8=-72$.

Since $12\times(-6)=-72$, our result checks.

2. $(4\,xyz^2)\times(-8\,x^2yz)$. Check when $x=1$, $y=2$, $z=2$.

3. $(2\,a^2bc^2d)\times(3\,ab^2c)$. Check when $a=1$, $b=1$, $c=1$, $d=1$.

4. $(-4\,a^2bc^2)\times(2\,ab)\times(-3\,ac)$. Check when $a=1$, $b=2$, $c=1$.

[Hint. Multiply the first two expressions together, then multiply what you get by the third expression.]

5. $(2\,abcd)\cdot(4\,ab)\cdot(ac)\cdot(bd)$. Check for $a=2$, $b=1$, $c=1$, $d=1$.

6. $(-x^ay^b)\cdot(x^my^n)$. Check for $x=2$, $y=2$, $a=2$, $b=2$, $m=1$, $n=1$.

7. Simplify $(-3\,a^2x)(5\,a^3x^2y)+(2\,m^2n)(-3\,n^3q)$.

8. Check your answer for Ex. 7, by using $a=1$, $x=1$, $y=2$, $m=2$, $n=1$, $q=2$.

9. Simplify

$$(4\,mnqs)(-3\,abqr)(2\,brs)-(2\,ghk)(-4\,qrs)(-3\,abc).$$

50. Raising a Monomial to a Power. We shall note first the following examples:

$$(xy)^2=(xy)\cdot(xy)=xyxy=xxyy=x^2y^2.$$
$$(xy)^3=(xy)\cdot(xy)\cdot(xy)=xxxyyy=x^3y^3.$$
$$(xy)^4=(xy)\cdot(xy)\cdot(xy)\cdot(xy)=xxxxyyyy=x^4y^4.$$
$$(xy)^5=\cdots=x^5y^5.$$

From these we infer the following general rule.

RULE FOR RAISING A PRODUCT TO A POWER. *To raise the product of two numbers to any power, raise the two numbers separately to that power and take their product.*

Stated as a formula, this becomes

Formula II. $$(xy)^m=x^my^m.$$

Illustrations:

$$(2\,x)^2=2^2\,x^2=4\,x^2.$$
$$(3\,y)^3=3^3\,y^3=27\,y^3.$$
$$(2\,xy)^2=(2\,x\cdot y)^2=(2\,x)^2y^2=2^2\,x^2y^2=4\,x^2y^2.$$

ORAL EXERCISES

State the result in each of the following exercises.

1. $(2\,x)^3$.
2. $(3\,m)^2$.
3. $(5\,s)^3$.
4. $(3\,ab)^2$.
5. $(2\,mn)^3$.
6. $(8\,abc)^2$.
7. $(3\,xyz)^3$.
8. $(2\,abc)^4$.
9. $(8\,pqrs)^2$.
10. $(-3\,ab)^3$.
11. $(-2\,mn)^4$.
12. $(4\,x^2)^2$.
13. $(-2\,x^2y^2)^3$.
14. $(7\,a^2bc)^2$.
15. $(8\,xyz)^2+(3\,mn)^3$.
16. $(-3\,xy)^3-(-2\,xy)^4+4(2\,xy)^2$.

17. In the figure are two squares, a side of the second one being twice as long as a side of the other. Show that the *area* of the second is *four* times that of the first.

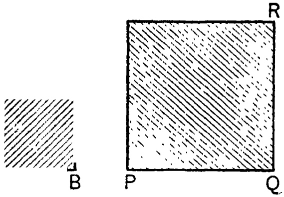


FIG. 19.

[HINT. Let $a =$ the length of side of the first. Then $2a =$ the length of side of the second. Now compare $(2a)^2$ with a^2.]

18. Show that if one cube has its edges each twice as long as the edges of a certain other cube, the *volume* of the one is *eight* times that of the other. See Ex. 17.

19. Compare the areas of two circles, one of which has a radius twice as great as the other. See Ex. 25, p. 22.

20. Compare the volumes of two spheres, one of which has a radius twice as great as the other. See Ex. 28, p. 23.

51. Multiplication of a Polynomial by a Monomial. In § 49 we saw how to multiply any monomial by another monomial. Let us now see how to multiply any *polynomial* by a monomial. If in arithmetic we wish to multiply 6 ft. and 2 in. by 3, we first multiply the 6 ft. by 3, getting 18 ft., then we multiply the 2 in. by 3, getting 6 in. The entire answer is 18 ft. and 6 in. This may be written as follows:

$$\begin{array}{r} 6 \text{ ft.} + 2 \text{ in.} \\ 3 \\ \hline 18 \text{ ft.} + 6 \text{ in.} \end{array} \quad \textit{Ans.}$$

In the same way, if in algebra we wish to multiply the polynomial $6a+2b$ by 3 we multiply each part separately and add the results, as indicated in the following scheme:

$$\begin{array}{r} 6a+2b \\ 3 \\ \hline 18a+6b \end{array} \quad \textit{Ans.}$$

Similarly, if we wish to multiply $a^2-6ab+8b^2$ by $2a^2b$ the work is arranged as follows:

$$\begin{array}{r} a^2-6ab+8b^2 \\ 2a^2b \\ \hline 2a^4b-12a^3b^2+16a^2b^3. \quad \textit{Ans.} \end{array}$$

From these illustrations, we have the following rule.

RULE FOR MULTIPLYING A POLYNOMIAL BY A MONOMIAL. *To multiply a polynomial by a monomial, multiply each term of the polynomial separately and combine results.*

This rule is stated in simple form in the following formula.

Formula III. $\quad a(b+c)=ab+ac.$

ORAL EXERCISES

Multiply

1. $a+b$ by 3. *Ans.* $3a+3b$.
2. $a-b$ by 4.
3. $a+5$ by -2.
4. $x+y$ by x.
5. x^2+y^2 by $-x^3$.
6. $ab+xy$ by ax.
7. $a+b-c$ by b.
8. $3x^2-3xy+5y$ by $-2xy$.
9. $a^2-10ab+15b^2$ by $4a^2b^2$.
10. Simplify $(-2ac+4ax)(-5acx)$.
11. Subtract $a(b+c)$ from $b(a+c)$.
12. Show that the relation $a(b+c)=ab+ac$ is illustrated by the figure below.

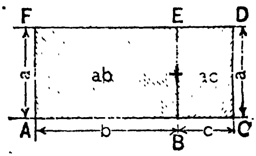


FIG. 20.

[HINT. ab is the area of the rectangle whose dimensions (length and breadth) are a and b.]

WRITTEN EXERCISES

Carry out the following indicated multiplications.

1. $mn(m-n+mn)$. Check when $m=1$, $n=2$.

SOLUTION. Multiplying in the way shown in § 51, the work appears as follows:

$$\begin{array}{r} m-n+mn \\ mn \\ \hline m^2n-mn^2+m^2n^2. \end{array} \quad \textit{Ans.}$$

CHECK. Suppose $m=2$ and $n=1$. Then $m-n+mn=2-1+2=3$, and $mn=2$ and the product of these is $3\times2=6$. At the same time, our answer (which is $m^2n-mn^2+m^2n^2$) becomes $4\times1-2\times1+4\times1=6$. Since both results are 6, the work checks.

2. $-2\,m(m+mn-n)$. Check when $m=1$, $n=2$.

3. $-6\,ab(-2\,a^2+4\,ab-3\,b^2)$. Check when $a=1$, $b=1$.

4. $(ab+bc+ac)abc$. Check when $a=2$, $b=2$, $c=1$.

5. $-3\,x^2(-2\,x^3+3\,x^2+4\,x)$. Check when $x=3$.

6. $r^3(-3\,r^3+4\,r^2+2\,r)$. Check when $r=3$.

7. $rs(1+2\,r-3\,s)$.

8. $3\,xy(-2\,xy^3+4\,x^2y^2-x^3y)$.

9. $8\,ab^3c^2(-2\,a^2bc^3-ab^4c^2+1)$.

10. $xy^2(x^3-2\,x^2y^2+y^3)$.

11. $(L^2-3\,L^3+4)2\,L^2$.

Simplify the following expressions.

12. $2(3\,a+4\,b)-2(2\,a-b)$.

13. $2\,x(3\,x-2\,y)+3\,x(x+2\,y)$.

14. $2\,b(b^2-b)-2\,b^2$.

15. $ab^3-ab(a^3+b^3)$. Check for $a=1$, $b=2$.

16. Show that $a(d+e)+b(d+e)+c(d+e)-e(a+b+c)=(a+b+c)d$.

For other exercises on this topic, see review exercises, p. **114**, *and Appendix*, p. 298.

52. Factoring an Expression. In the preceding exercises certain expressions were given us that had been separated into their factors, and we were asked to multiply the factors together. Thus, in Ex. 1, we were given the expression $mn(m-n+mn)$ and we multiplied this, getting the product $m^2n-mn^2+m^2n^2$.

Suppose now we try to reverse this process; that is, suppose we start with a product itself and ask what the factors are which when multiplied together give that product. We can often answer such questions in the manner illustrated below.

EXAMPLE 1. Factor the expression $ab+ac-ad$.

SOLUTION. The letter a is here contained as a factor in every term. Hence we may write

$$ab+ac-ad=a(b+c-d). \quad \textit{Ans.}$$

CHECK. When we simplify the answer by multiplying it out (as in § 51) we get $ab+ac-ad$.

EXAMPLE 2. Factor the expression $a^2x+ax^2+a^2x^2$.

SOLUTION. Here ax is a factor of every term. Hence, taking it out,

$$a^2x+ax^2+a^2x^2=ax(a+x+ax). \quad \textit{Ans.}$$

CHECK. Multiplying out in $ax\ (a+x+ax)$ gives $a^2x+ax^2+a^2x^2$.

EXAMPLE 3. Factor $3\,x^3y^3-3\,x^2y^2+12\,xy$.

SOLUTION. Here $3\,xy$ is a factor of every term. Hence,

$$3\,x^3y^3-3\,x^2y^2+12\,xy=3\,xy(x^2y^2-xy+4). \quad \textit{Ans.}$$

WRITTEN EXERCISES

Factor each of the following expressions, and check your answer in each case.

1. $3\,x+3\,y$. *Ans.* $3(x+y)$.

2. $mx+my$.

3. $mx-my$.

4. x^2+x.

5. x^2y+xy^2.

[HINT. xy is a factor in each term.]

6. $6\,a^2+16\,a$.

7. $2a^3+4a^3b+6a^2x$.
8. $2m+4n-8q$.
9. $3x^2+12xy+9x$.
10. $ab^2+b^3-b^2c$.
11. $4x^3y-6x^2y^2+10xy^3$.
12. $10a^2c^2x-20a^2cx^2$.
13. $16a^2b^3c^4-24a^3b^2c^3+32a^3b^4c^3$.
14. $14a^2mn^2-21a^3m^2n^3-49a^4mn^2$.

For further exercises on this topic, see the review exercises, p. 114, and Appendix, p. 298.

53. Multiplication of a Polynomial by a Polynomial. In multiplying a polynomial by a polynomial we multiply the multiplicand by each of the monomials in the multiplier, and then combine these partial results. For example, in multiplying $2x-3y$ by $3x+4y$ the work is as follows:

$$\begin{array}{rl}
 & 2x-3y \\
 & \underline{3x+4y} \\
\text{Multiplying } 2x-3y \text{ by } 3x \text{ gives} & 6x^2-9xy \\
\text{Multiplying } 2x-3y \text{ by } 4y \text{ gives} & \underline{\phantom{6x^2-{}}8xy-12y^2} \\
\text{Adding gives} & 6x^2-xy-12y^2 \quad \textit{Ans.}
\end{array}$$

NOTE. In this work the expressions $6x^2-9xy$ and $8xy-12y^2$ are called *partial products*. The adding of the partial products always gives the answer.

Again, let us multiply $x-y+3z$ by $2x+3y-z$. The work, which should be examined carefully, follows.

$$\begin{array}{rl}
 & x-y+3z \\
 & \underline{2x+3y-z} \\
\text{Multiplying by } 2x \text{ gives} & 2x^2-2xy+6xz \\
\text{Multiplying by } 3y \text{ gives} & \phantom{2x^2-{}}3xy-3y^2+9yz \\
\text{Multiplying by } -z \text{ gives} & \underline{-xz+yz-3z^2} \\
 & 2x^2+xy+5xz-3y^2+10yz-3z^2.
\end{array}$$

CHECK. Suppose $x=1$, $y=1$, $z=1$. Then $x-y+3z=1-1+3=3$, and $2x+3y-z=2+3-1=4$, and the product of these is $3\times4=12$. At the same time, when $x=1$, $y=1$, $z=1$, our answer (which is $2x^2+xy+5xz-3y^2+10yz-3z^2$) becomes $2+1+5-3+10-3=12$. Since both are 12, the work checks.

WRITTEN EXERCISES

Carry out each of the following indicated multiplications.

1. $(x+r)(x+r)$. Check when $x=2$, $r=1$.
2. $(x-r)(x-r)$. Check when $x=1$, $r=2$.
3. $(x+r)(x-r)$. Check when $x=2$, $r=2$.
4. $(a-2)(a-2)$. Check when $a=3$.
5. $(a+2)(a-2)$. Check when $a=4$.
6. $(m+n)(m+n)$. Check when $m=2$, $n=2$.
7. $(m^2+n^2)(m^2-n^2)$. Check when $m=2$, $n=2$.
8. $(m^2-n^2)(m^2-n^2)$. Check when $m=1$, $n=2$.
9. $(ab+c)(ab-c)$. Check when $a=1$, $b=2$, $c=1$.
10. $(ab-5)(ab+4)$. Check when $a=2$, $b=3$.
11. $(2r+7)(3r+5)$.
12. $(3r+s-2t)(r-5s)$.
13. $(x^2y^2-6)(x^2y^2-2)$.
14. $(3a^2-2)(4a+1)$.
15. $(5y^2+3z^2)(2y-z)$.
16. $(4a-10b+1)(2a-b+2)$.
17. $(11+a+b^2)(4-5a-b^2)$.
18. $(8r^2-5r+1)(3r^2+2r-2)$.
19. $(3x^2-2y^2)(x^2-3y^2)$.
20. $(A+B-C)(A-B+C)$.

APPLIED PROBLEMS

21. If the side of a square is represented by $2x+3$, what represents its area?

SOLUTION. The area will be represented by $(2x+3)^2$, which, when multiplied out, becomes $4x^2+12x+9$. *Ans.*

22. If the side of a square is represented by $5x-2$, what represents its area?

23. If the edge of a cube is represented by $3x+2$, what represents its volume?

24. If the edge of a cube is represented by $2x+y$, what represents its volume?

25. If the dimensions (length and breadth) of a rectangle are represented by $x+2$ and $x-1$, what represents the area?

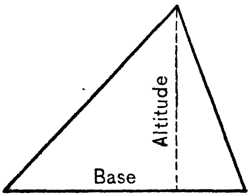


FIG. 21.

26. We know (see Example 18, page 21) that the area of a triangle equals one half the product of its base times its altitude (or height). What represents the areas of the triangles having the following bases and altitudes?

	BASE	ALTITUDE
(*a*)	$2x+3$	$x-5$
(*b*)	$5a+6$	$a-2$
(*c*)	$3r-s$	$r-s$

27. What is the area of the triangle in Ex. 26 (*c*) in case $r=3$, $s=2$?

28. A trapezoid is a four-sided figure whose upper and lower sides (called bases) are parallel. The area of a trapezoid is equal to one half the sum of its bases multiplied by its altitude. What formula, therefore, represents the area of the trapezoid whose bases are a and b and whose altitude is h?

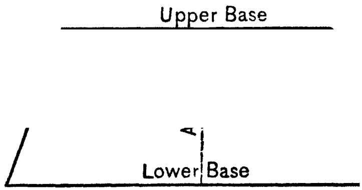


FIG. 22. — TRAPEZOID.

29. What represents the area of the trapezoid whose bases are $2x$ and $3y$ and whose altitude is $x+y$?

30. What is the area of the trapezoid in Ex. 29 if $x=2$, $y=4$?

31. If the radius of a circle is represented by $x+3$, what represents its area?

54. Further Study of Factoring. We saw in § 52 how such an expression as $3\,ax^3+6\,x^2+12\,bx$ could be factored. In fact, since $3\,x$ is a factor in each term, the answer here is $3\,x(ax^2+2\,x+4\,b)$. Note that in all such cases the factor common to the terms (in this case $3\,x$), is a *monomial.* We shall now consider some examples in factoring in which the common term is *not* a monomial.

EXAMPLE 1. Factor the expression $2(a+b)+x(a+b)$.

SOLUTION. Since $a+b$ is a common factor of both terms, as in § 52 we have

$$2(a+b)+x(a+b)=(a+b)(2+x). \quad \textit{Ans.}$$

CHECK. When we multiply $a+b$ by $2+x$ (in the way shown in § 53) we get $2\,a+2\,b+xa+xb$, and this is the same as $2(a+b)+x(a+b)$.

EXAMPLE 2. Factor the expression $x(x+y)+y(x+y)+2(x+y)$.

SOLUTION. Here $x+y$ is a common factor of each term. Hence

$$x(x+y)+y(x+y)+2(x+y)=(x+y)(x+y+2). \quad \textit{Ans.}$$

CHECK. Multiplying $x+y$ by $x+y+2$ as in § 53 gives

$$x^2+2\,xy+y^2+2\,x+2\,y.$$

But this is also the result one gets when he performs the multiplications and simplifies the expression

$$x(x+y)+y(x+y)+2(x+y).$$

EXAMPLE 3. Factor the expression $ax+ay+bx+by$.

SOLUTION. First observe that the expression may be written in the form $(ax+ay)+(bx+by)$, thus grouping terms having a common factor. This form may be changed to $a(x+y)+b(x+y)$. Now proceed as in Examples 1 and 2, and obtain the answer $(x+y)(a+b)$.

EXAMPLE 4. Factor $cx+y-dy+cy-dx+x$.

SOLUTION. By rearranging the terms, the expression may be written in the form

$$cx-dx+x+cy-dy+y=(c-d+1)x+(c-d+1)y=(c-d+1)(x+y). \quad \textit{Ans.}$$

EXERCISES

Factor each of the following expressions.

1. $a(x-y)+b(x-y)$.
2. $3x(2y-3z)+4y(2y-3z)$.
3. $m(a+b-c)+n(a+b-c)+q(a+b-c)$.
4. $6a(a-b)+5b(a-b)-2(a-b)$.
5. $a(a-b)+3(b-a)$.

[HINT. Write $3(b-a)$ in the form $-3(a-b)$.]

6. $am-an+mx-nx$.

[HINT. Proceed as in Example 3, § 54.]

7. $pq-px-rq+rx$.
8. $ay-by-ab+b^2$.
9. $2x-y+4x^2-2xy$.
10. $y^2-4y+xy-4x$.
11. $1-p+q-pq$.
12. $x^2+x+x+1$.
13. $x^2-x-a+ax$.

55. Arrangement of Terms before Multiplying. We have already seen in § 33 how the addition and subtraction of polynomials is best carried out by first arranging them in the ascending or descending powers of some one letter. This is also true in the multiplication of two polynomials. For example, in multiplying $x-1-3x^3+2x^4$ by $2+x$, we first arrange both in descending powers of x, thus giving them the forms $2x^4-3x^3+x-1$ and $x+2$. Then we multiply (in the manner explained in § 53), the work appearing as below:

$$\begin{array}{rrrrrr}
2x^4 & -3x^3 & & +x & -1 & \\
x & +2 & & & & \\
\hline
2x^5 & -3x^4 & & +x^2 & -x & \\
 & 4x^4 & -6x^3 & & +2x & -2 \\
\hline
2x^5 & +x^4 & -6x^3 & +x^2 & +x & -2.
\end{array} \quad \textit{Ans.}$$

WRITTEN EXERCISES

1. Multiply $x-1-3x^3+2x^4$ by $2+x$.

[HINT. First arrange the polynomials in *ascending* powers of x. Is your result the same as was obtained in § 55?]

Carry out each of the following indicated multiplications, first rearranging where desirable.

2. $(3a^2-2a-1)(a-1)$. Check for $a=2$.

3. $(x-3+x^2)(2+x)$. Check for $x=2$.

4. $(3a^2+2a-4)(5-a)$.

5. $(5n-4+6n^2)(8+n^2-4n)$.

6. $(2+3x^2-x+x^3)(x^2-2x+4)$.

7. $(x^2+2xy+y^2)(x+y)$.

[HINT. Here both polynomials are already arranged according to descending powers of x.]

8. $(a^2-ab+b^2)(a+b)$.

9. $(a^2+ab+b^2)(a-b)$.

10. $(m^2-mn+n^2)(m^2+mn+n^2)$. Check for $m=1$, $n=1$.

11. $(8r^2-2s^2+4rs)(2rs+3s^2+4r^2)$.

[HINT. Arrange in descending powers of r.]

12. $(A^2+B^2-2AB)(A^2+B^2+2AB)$.

13. $(x^2y^2+xy+1)(1-xy+x^2y^2)$.

14. $(4x^3-3x^2y+5xy^2-6y^3)(5x+6y)$.

15. Expand $(x+r)^2$. [HINT. $(x+r)^2$ means $(x+r)(x+r)$.]

Expand each of the following expressions.

16. $(a+2)^2$.

17. $(2a+3b)^2$.

18. $(a-b)^2$.

19. $(a-2)^2$.

20. $(2a-3b)^2$.

21. $(3x+4)^2$.

22. $(2a-5)^2$.

23. $(x^2+2x+1)^2$. Check for $x=2$.

24. $(x+1)^3$. [HINT. Find $(x+1)^2$ and multiply it by $x+1$.]

25. $(x+1)^4$.

26. $(x+1)^2(x+2)^2$.

27. $(a+b+c+d)^2$.

28. $(x-y)(x+y)(x^4+x^2y^2+y^4)$.

WRITTEN EXERCISES

Combinations of Addition, Subtraction, and Multiplication

1. Simplify $a^2+a(b-a)-b(3\,b-a)$.

Solution. This expression means the sum of a^2, $a(b-a)$ and $-b(3\,b-a)$. Now, $a(b-a)=ab-a^2$ and $-b(3\,b-a)=-3\,b^2+ab$. Therefore, writing the terms in their order with proper signs, the expression becomes $a^2+ab-a^2-3\,b^2+ab=2\,ab-3\,b^2$. *Ans.*

Simplify.

2. $x^2+x(y-x)$.

3. $x^2-x(x-y)$.

4. $4-2(a-3)$.

5. $x^2-y^2-(x-y)^2$.

6. $(a-b)^2-3(a^2+b^2)$.

7. $(x+y)^2-(x-y)^2$.

8. $(3+a)(3-a)-(3-a)^2$.

9. $a^3-b^3-3\,ab(a-b)$.

10. $-2\,xy(x-y)+3\,xy(y-x)$.

11. $(3\,a-2)(2\,a-3)-6(a-2)(a-1)$.

12. $8\,x^3-(4\,x^2-2\,xy+y^2)(2\,x+y)$.

13. $(3\,r-1)(r+2)-3\,r(r+3)+2(r+1)$.

14. $(x-y)^2-2(x^2-y^2)-2\,x(-x-y)-4\,y^2$.

15. $(x+y)^2(x-y)^2-(x^2+y^2)^2$.

16. $n^4+(m^2-mn+n^2)(m^2+n^2)-(m^3-n^3)(m+2\,n)$.

17. $(x+y)(x^2-xy+y^2)+(x-y)(x^2+xy+y^2)$.

56. Further Study of the Equation. We frequently meet with equations containing such multiplications as those just studied. For example, consider the equation $5\,x+x(x-3)=12+x^2$. The root (or value of the unknown number x) is found as follows:

$$5\,x+x(x-3)=12+x^2.$$

Carrying out the indicated multiplication gives

$$5\,x+x^2-3\,x=12+x^2.$$

Canceling x^2 from both sides gives

$$5\,x-3\,x=12, \quad \text{or} \quad 2\,x=12.$$

Therefore,

$$x=6. \quad \textit{Ans.}$$

CHECK. When $x=6$ the first member of the given equation becomes $5\times6+6\times3=30+18=48$, while the second member becomes $12+36=48$. Since both have the same value, namely 48, the root (or $x=6$) is correct.

WRITTEN EXERCISES

Solve and check your answer in each of the following equations.

1. $3(x+4)=18.$
2. $2(x+5)+3(x-6)=12.$
3. $5(x+4)+2(x-3)=4(x-1).$
4. $(x+2)(x+3)=x(x+3)+10.$
5. $(x-4)(x+5)=x(x-2).$
6. $(r-3)(r-5)=(r+4)(r-2)-7.$
7. $(y+1)(y-2)=(y-3)(y-4).$
8. $(s+3)(s+8)-s(s+6)=0.$
9. $(x^2-x-1)(2\,x+4)=2\,x^2(x+1)-7(x-1).$
10. $x^2-(2\,x+3)(2\,x-3)+(2\,x-3)^2=(x+9)(x-2)-2.$
11. $3(4-x)^2-2(x+3)=(2\,x-3)^2-(x+2)(x-2)+1.$
12. $3\,x^2-\{5\,x-[4-(x-1)(2\,x-3)-7\,x]+(x-3)^2\}=0.$
13. $5\,x+1-2\{2\,x-3[x-(x+1)(x+3)]-3(x+2)^2\}=0.$

For further exercises on this topic, see Appendix, p. **299.**

EXERCISES — APPLIED PROBLEMS

These problems resemble those on pp. 76, 77, yet they are different because they cannot be worked without a knowledge of the principles of *multiplication* studied in this Chapter.

1. Divide 29 into two parts so that three times the one part added to five times the other part shall equal 105.

SOLUTION. Let x=one part. Then the other part $=29-x$.
The problem then says that

$$3x+5(29-x)=105.$$

Multiplying (see § 51), this becomes

$$3x+145-5x=105.$$

Transposing gives

$$-2x=-40.$$

Dividing both sides by -2 gives

$$x=20.$$

The one part is therefore 20, and the other is $29-20$, or 9. *Ans.*

2. The difference between the ages of a father and his son is 45 years and in a year's time the father will be six times as old as the son. How old is each?

[HINT. Let $x=$ the number of years in the son's age. Then $x+45=$ the number of years in the father's age. A year from now their ages will be $x+1$ and $x+46$.]

3. A father is four times as old as his son; five years ago he was five times as old. How old is each?

4. A has $37 and B has $11. How much must A give B in order that A may then have twice as much as B?

5. A box of 28 chocolates was divided between 6 boys and 4 girls, each girl receiving 2 more than a boy. How many did each girl receive?

6. In buying material for a dress a lady bought 8 yards of silk and 4 yards of trimming, paying 90 cents more per

yard for the trimming than for the silk. The bill was \$18.00. How much did each cost per yard?

[HINT. Work in cents.]

7. A bag contains 52 silver coins, some of them quarter-dollars and the rest dimes. The value of the whole is \$10.00. How many coins of each kind are there?

8. A man has \$100 in one bank and \$25 in another. If he has \$125 more to deposit, how should he divide it between the two banks so that the first account may become four times as great as the second?

[HINT. Compare Ex. 44, p. 26.]

9. A father made this offer to his son, "For every day on which your standing in algebra is as high as 80% I will give you a nickel, but on every day that your standing falls below 80% you are to give me a penny." At the end of 12 days the son had 18 cents. For how many days was his standing above 80%?

10. In the figure the shaded part, or border, represents the area between an inner square (lettered *abcd*) and an outer square (lettered *ABCD*). If this border contains 32 square feet and is everywhere 1 foot wide, how long must one of the sides of the inner square be?

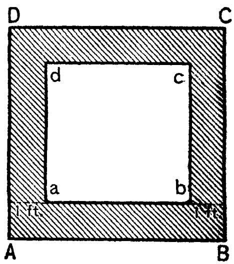


FIG. 23.

[HINT. Let $x=$ the length of one of the sides of the inner square. Then $x^2=$ the *area* of the inner square, and $x^2+32=$ the area of the inner square and border.

But the area of the inner square and border is the same as that of the outer square, and this is $(x+2)^2$, since each side of the outer square $x+1+1$, or $x+2$.]

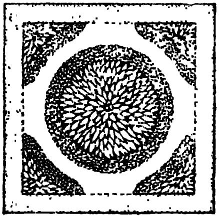


FIG. 24.

11. It is desired to lay out a square flower garden with a gravel walk around it, the gravel walk to be 2 feet wide and 6 inches deep. How long can each side of the garden be if only 5 loads of gravel are to be used, it being given that a load contains 1 cubic yard?

[HINT. First find how much area the gravel will cover when laid 6 inches deep. Then proceed as in Ex. 10.]

12. Suppose that the garden mentioned in Ex. 11 is *rectangular*, with one of the long sides measuring 2 feet more than a short side, and suppose as before that the gravel border is to be 6 inches deep and 2 feet wide, ten loads of gravel to be used. Show that in this case the equation for the shorter side, x, becomes $x(x+2)+540=(x+4)(x+6)$. Whence, find the dimensions (length and breadth) of the rectangle.

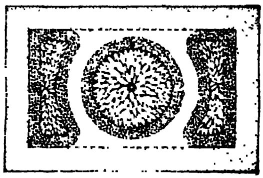


FIG. 25.

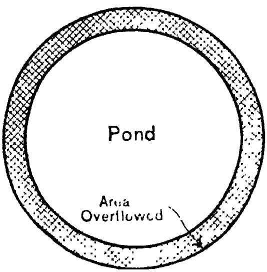


FIG. 26.

13. During a spring freshet a certain circular pond overflowed its banks so that an acre of ground all around was inundated (under water). It was found afterwards that the overflow had carried the water back from the bank about 20 feet on all sides. What was the diameter of the pond in ordinary weather?

[HINT. See Ex. 25 page 22. Take $\pi=3\frac{1}{7}$. Remember the diameter = twice the radius.]

PART II. SPECIAL CASES IN MULTIPLICATION AND FACTORING

57. Product of Two Binomials. We have to multiply binomials together so often in algebra that it is desirable to know how to get the answer quickly by mere inspection. This is possible in many cases.

Thus, in multiplying $x+3$
by $x+4$

$$\begin{array}{r} x+3 \\ x+4 \\ \hline x^2+3\,x \\ 4\,x+12 \\ \hline x^2+7\,x+12 \end{array}$$

the answer could easily be obtained in the following way: First, square the x, giving x^2, then add (mentally) the 3 and the 4, giving 7, and multiply this by x, giving the $7\,x$ of the answer. Finally, multiply (mentally) the 3 by the 4, giving the 12 which appears at the end. By adding the x^2, the $7\,x$ and the 12 obtained in this way, we have the answer.

Let us take another illustration. Suppose we wish to multiply $x+2$ by $x+3$. Reasoning as before, the answer will be $x^2+(2+3)x+2\times 3$, or $x^2+5\,x+6$. Check this yourself by multiplying $x+2$ by $x+3$ in the long way.

Similarly, we can shorten the work of multiplying even when negative numbers occur. For example, in multiplying $x-2$ by $x+1$ the answer will be $x^2+(-2+1)x+(-2)\times 1$, or x^2-x-2.

Other illustrations follow and should be carefully examined.

1. $(x+7)(x-2)=x^2+(7-2)x+7(-2)=x^2+5\,x-14$. *Ans.*

2. $(x-8)(x+3)=x^2+(-8+3)x+(-8)3=x^2-5\,x-24$. *Ans.*

3. $(x-6)(x-4)=x^2+(-6-4)x+(-6)(-4)$
$=x^2-10\,x+24$. *Ans.*

When stated concisely, all such cases of multiplication come under the following formula.

Formula IV. $(x+m)(x+n)=x^2+(m+n)x+mn.$

It is to be remembered here that m and n represent *any* numbers, positive or negative.

ORAL EXERCISES

State the answer by inspection in each of the following exercises.

1. $(x+3)(x-4)$.

2. $(x-3)(x-4)$.

3. $(r-3)(r+4)$.

4. $(r+3)(r+4)$.

5. $(a+6)(a-8)$.

6. $(a+12)(a+4)$.

7. $(r-11)(r+10)$.

8. $(s+12)(s+2)$.

9. $(x-9)(x+6)$.

10. $(x-3)(x-10)$.

11. $(x-12)(x-2)$.

12. $(y+1)(y-8)$.

13. $(y-1)(y+8)$.

14. $(y+8)(y-1)$.

15. $(z+11)(z-4)$.

16. $(A+3)(A+7)$.

17. $(n-4)(n+8)$.

18. $(t+5)(t-8)$.

19. $(R+1)(R-6)$.

20. $(x-12)(x+8)$.

21. $(x+16)(x-9)$.

22. $(S+6)(S-7)$.

23. $(ab+2)(ab+6)$.

[HINT. By Formula IV the result is $(ab)^2+8\,ab+12$. This may be written (see Formula II) $a^2b^2+8\,ab+12$. *Ans.*]

24. $(ab-6)(ab-5)$.

25. $(c^2-3)(c^2-4)$.

HINT. Use Formulas IV and I.]

26. $(c^2-2)(c^2+3)$.

27. $(a^3+4)(a^3+6)$.

28. $(a^2b^2-1)(a^2b^2+3)$.

29. $(2\,a+3)(2\,a-7)$.

[HINT. The result, by Formula IV, is $(2\,a)^2-4(2\,a)-21$. This may be written

$4\,a^2-8\,a-21$. *Ans.*]

30. $(3\,x-4)(3\,x+2)$.

31. $(4\,x+3)(4\,x+1)$.

32. $(6x-5)(6x+7)$.
33. $(8r+5)(8r-6)$.
34. $(2S+6)(2S+10)$.
35. $(6V-3)(6V-6)$.
36. $(H+2)(H-19)$.
37. $(a-10b)(a+4b)$.

[HINT. The result, by Formula IV, is
$a^2+(-10b+4b)a+(-10b)(4b)$.
This reduces to $a^2-6ab-40b^2$. *Ans.*]

38. $(r+7s)(r-6s)$.
39. $(x-2r)(x+3r)$.
40. $(t+10z)(t-z)$.
41. $(2t^2+3z)(2t^2+4z)$.
42. $(3a^2-5y)(3a^2+7y)$.
43. $(x+\frac{1}{2})(x+\frac{1}{4})$.
44. $(x-\frac{2}{3})(x+\frac{1}{3})$.
45. $(a+\frac{5}{6})(a-\frac{1}{2})$.
46. $12 \cdot 13$

SOLUTION. $12 \cdot 13=(10+2)(10+3)$ which, by Formula IV, is $10^2+(2+3)10+2 \cdot 3=100+50+6=156$. *Ans.*

47. $22 \cdot 21$
48. $18 \cdot 16$
49. $28 \cdot 33$

[HINT. Write as $(30-2)(30+3)$.]

50. $31 \cdot 28$

For further exercises on this topic, see the review exercises, p 114, and Appendix, p. 299.

58. The Factoring of a Trinomial. In § 57 and in the exercises that follow it we started every time with two given binomials, such as $x+2$ and $x+3$, and we saw how to get their product very easily; in fact, we learned to do it mentally. In every case the product turned out to be a *trinomial* beginning with the square of the common letter. Thus, in multiplying $x+2$ by $x+3$ the result is the trinomial x^2+5x+6, which begins with x^2.

Suppose now we try to turn this idea around. Let us start with a given trinomial, such as x^2+7x+6, and inquire what two binomials give this when multiplied together. In other words, let us try to find the two binomial *factors* of x^2+7x+6. This can be done as follows. We think of the two numbers whose *sum* is 7 and whose *product* is 6.

The numbers that do this are 6 and 1. Then $x+6$ and $x+1$ must be the factors we want, for when we multiply $(x+6)$ by $(x+1)$, using Formula IV, we get (as we should) our trinomial, x^2+7x+6.

Similarly, in order to find the factors of x^2+2x-8 we think of the two numbers whose *sum* is 2 and whose *product* is -8. The numbers are seen to be 4 and -2. Therefore, the desired factors are $x+4$ and $x-2$. *Ans.*

NOTE. It is to be observed that in all such problems, the first term of each of the desired factors is simply x; that is, it is the *square root* of the first term of the given trinomial.

WRITTEN EXERCISES

Find the two binomial factors of each of the following expressions.

1. $x^2+7x+12$.

[HINT. Here we need the two numbers whose *sum* is 7 and whose *product* is 12.]

2. x^2-5x+6. [HINT. Proceed as in Ex. 1.]

3. a^2+6a+8.

4. a^2-6a+8.

5. $x^2-7x+12$.

6. x^2-x-12.

7. r^2+5r+4.

8. $y^2-4y-12$.

9. $m^2-6m-16$.

10. z^2-6z+5.

11. $s^2-4s-21$.

12. x^2-x-30.

13. $z^2+3z-18$.

14. b^2+b-42.

15. $c^2+7c-18$.

16. $s^2-6s-40$.

17. s^2+6s+5.

18. $k^2-8k+15$.

19. $k^2-10k+9$.

20. $z^2+2z-15$.

21. $x^2+xy-56y^2$.

22. $x^2-5xy-36y^2$.

23. $t^2+12t-108$.

24. $x^2y^2+6xy-16$.

25. $a^2b^2-21ab-72$.

26. $12+7a+a^2$.

[HINT. Rearrange the terms.]

27. $33+14z+z^2$.

28. $14+9k+k^2$.

29. $x^2-\frac{3}{2}x+\frac{1}{2}$.

For further exercises on this topic, see the Appendix, p. 299.

59. The Square of the Sum of Two Numbers. Suppose we have any two numbers, a and b, and let us form their sum, $a+b$. Then $(a+b)^2$ means $(a+b) \cdot (a+b)$ and when multiplied out it looks as follows:

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ ab+b^2 \\ a^2+2\,ab+b^2 \end{array}$$

Thus we have the following formula:

Formula V. $\boldsymbol{(a+b)^2=a^2+2\,ab+b^2}$.

Expressed in words, this says that *the square of the sum of any two numbers is equal to the square of the first number plus twice the product of the two plus the square of the second number.*

For example,

$$(r+6)^2=r^2+2(r \cdot 6)+6^2=r^2+12\,r+36.$$

$$(2\,a+3\,b)^2=(2\,a)^2+2(2\,a \cdot 3\,b)+(3\,b)^2=4\,a^2+12\,ab+9\,b^2.$$

Check each of these examples by multiplying the long way, as in § 53.

NOTE. Formula V is really a special case of Formula IV, for if in that formula we use a for x, b for m, and b for n, we obtain

$$(a+b) \cdot (a+b)=a^2+(b+b)a+b^2,$$

which is the same as $(a+b)^2=a^2+2\,ab+b^2$.

60. The Square of the Difference of two Numbers. Suppose we have any two numbers, a and b, and that we form their difference, $a-b$. Then $(a-b)^2$ means $(a-b) \cdot (a-b)$ and when multiplied out this looks as follows:

$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ -ab+b^2 \\ a^2-2\,ab+b^2 \end{array}$$

Thus we have the following formula.

Formula VI. $(a-b)^2=a^2-2ab+b^2$.

Expressed in words, this says that *the square of the difference of any two numbers is equal to the square of the first number minus twice the product of the two plus the square of the second number.*

For example,

$$(r-6)^2=r^2-2(r\cdot 6)+6^2=r^2-12r+36.$$
$$(3x-2b)^2=(3x)^2-2(3x\cdot 2b)+(2b)^2=9x^2-12bx+4b^2.$$

NOTE. Formula VI is a special case of Formula IV, for if we use a for x, $-b$ for m, and $-b$ for n in Formula IV we obtain $(a-b)\cdot(a-b)=a^2+(-b-b)a+b^2$ which is the same as $(a-b)^2=a^2-2ab+b^2$.

61. Applications of Formulas V and VI. The value of Formulas V and VI lies in the fact that they enable us to read *by inspection* the square of any binomial in algebra. For example, $(x+3)^2$ is immediately read by Formula V to be $x^2+2(3x)+3^2$, which reduces to x^2+6x+9. Likewise, $(2x-3)^2$ is read by Formula VI to be $(2x)^2-2(2x\cdot 3)+3^2$, which reduces to $x^2-12x+9$.

ORAL EXERCISES

Give by inspection the expanded values of each of the following expressions.

1. $(x+2)^2$.

2. $(x-2)^2$.

3. $(x+8)^2$.

4. $(x-8)^2$.

5. $(c+3)^2$.

6. $(r+7)^2$.

7. $(2x+1)^2$.

8. $(2x+3)^2$.

9. $(3x-2)^2$.

10. $(3x-4)^2$.

11. $(4+x)^2$.

12. $(4-x)^2$.

13. $(m+n)^2$.

14 $(m-n)^2$.

15. $(a+2b)^2$.

16. $(a-2b)^2$.

17. $(2a-b)^2$.

18. $(2r-3s)^2$.

19. $(xy+1)^2$.

20. $(4xy+1)^2$.

21. $(4xy-1)^2$.

22. $(ab+cd)^2$. **26.** $(r+\frac{1}{2})^2$. **30.** $(m-\frac{2}{3}n)^2$.

23. $(2ab-3cd)^2$. **27.** $(r-\frac{3}{4})^2$. **31.** $(2p+\frac{1}{2}q)^2$.

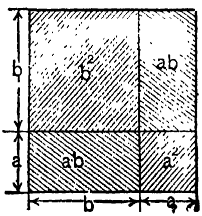
24. $(x+\frac{2}{3})^2$. **28.** $(a+\frac{1}{6}b)^2$. **32.** $(3m-\frac{1}{2}n)^2$.

25. $(x-\frac{2}{3})^2$. **29.** $(x-\frac{1}{2}y)^2$.

For further exercises on this topic see the review exercises, p. 114, *and Appendix, p.* 299.

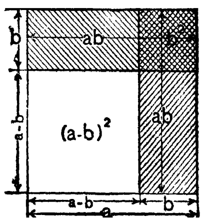
62. Geometric Meaning of Formulas V and VI. The expression $(a+b)^2$ means the *area* of the square whose side is $(a+b)$, and this area may be broken up into four parts (shaded in Fig. 27). Now, one of these parts (lower right-hand corner) is seen to be a square whose side is a, and hence this part has an area equal to a^2. Similarly another part has an area of b^2, while the remaining two parts are rectangles each having the area ab. The sum of the four parts is therefore, $a^2+b^2+ab+ab$, which may be written $a^2+2ab+b^2$. Thus, the figure illustrates how $(a+b)^2 = a^2+2ab+b^2$; that is, it illustrates Formula V.

While it is not so easy to illustrate the meaning of Formula VI, a careful examination of Fig. 28 will show that it does so. Note that the part $(a-b)^2$ is equal to the area of the whole square, or a^2, minus the area of the shaded border. But the shaded border is equal to $ab+ab-b^2$, or $2ab-b^2$. Thus $(a-b)^2=a^2-(2ab-b^2)$, which reduces to $a^2-2ab+b^2$, or Formula VI.



$(a+b)^2=a^2+2ab+b^2$

FIG. 27.



$(a-b)^2=a^2-2ab+b^2$

FIG. 28.

63. Trinomials That Are Perfect Squares. We have seen in §§ 59, 60 that the square of a binomial is always a trinomial. Thus,

$$(2x+y)^2=(2x)^2+2(2x\cdot y)+y^2=4x^2+4xy+y^2. \qquad \text{(Formula V)}$$

$$(x-3y)^2=x^2-2(x\cdot 3y)+(3y)^2=x^2-6xy+9y^2. \qquad \text{(Formula VI)}$$

A trinomial obtained in this way is called a ***trinomial square***, or a ***perfect square***. It must be carefully observed that not all trinomials are perfect squares. The essentials for this are that *two of the terms must be squares and positive, while the remaining term* (usually the middle one) *must be equal to twice the product of the square roots of the square terms*. This will now be illustrated by means of several examples.

EXAMPLE 1. x^2+6x+9 is a perfect square because its terms x^2 and 9 are squares and positive, while its remaining term, $6x$, is equal to $2\cdot\sqrt{x^2}\cdot\sqrt{9}$. Moreover, since the sign of the term $6x$ in the given trinomial is +, we know by Formula V that we here have the square of the *sum* of two numbers. In fact, what we have is $(x+3)^2$.

EXAMPLE 2. $m^2-4mn+4n^2$ is a perfect square because its terms m^2 and $4n^2$ are squares and positive, while its remaining term, $4mn$, is equal to $2\cdot\sqrt{m^2}\cdot\sqrt{4n^2}$. Moreover, since the sign of $4mn$ in the given trinomial is $-$, we know from Formula VI that we here have the square of the *difference* of two numbers. It is, in fact, $(m-2n)^2$.

EXAMPLE 3. $9x^2-10x+4$ is *not* a perfect square since the term $10x$ does not equal $2\cdot\sqrt{9x^2}\cdot\sqrt{4}$.

EXAMPLE 4. $4x^2-12x-9$ is *not* a perfect square since its last term is negative instead of positive.

ORAL EXERCISES

In each of the following exercises tell whether the given trinomial is a perfect square, and if so whether it is the square of the sum or of the difference of two numbers.

1. $x^2+2\,xy+y^2$.
2. $a^2-6\,a+9$.
3. $b^2-14\,b+49$.
4. $r^2+10\,rs-s^2$.
5. $m^2+8-4\,m$.
6. $t^2-16\,t+64$.
7. $4\,x^2-8\,x+4$.
8. $x^2-2\,xy-y^2$.
9. $16\,m^2+12\,mn+9\,n^2$.
10. $a^2+2\,a+1$.
11. $9\,b^2+6\,b+3$.
12. $25\,z^2-10\,z-1$.
13. $36\,a^2+12\,ab+b^2$.
14. $a^2b^2-2\,ab+4$.
15. $121\,a^2x^2-22\,ax+1$.
16. $4\,r^2+1+44\,r$.
17. $8\,r^2-16\,r+4$.
18. $144\,x^2-24\,x+1$.
19. $81\,x^2+36\,x+4$.
20. $14\,a^2+28\,ab+9\,b^2$.

WRITTEN EXERCISES

Supply the missing term in each of the following expressions so as to make a perfect square of the trinomial.

1. $x^2+6\,x+(\quad)$.

SOLUTION. Since we may write $6\,x=2\cdot\sqrt{x^2}\cdot\sqrt{9}$, we see that the missing term must be 9. *Ans.*

CHECK. With 9 supplied for the missing term we find (as in § 63) that the given trinomial is a perfect square.

2. $a^2+4\,a+(\quad)$.
3. $1-2\,a+(\quad)$.
4. $a^2b^2+(\quad)+1$.

SOLUTION. The missing term must be twice the product of the square roots of the other terms; that is, it must be $2\cdot\sqrt{a^2b^2}\cdot\sqrt{1}$ which reduces to $2\,ab$. *Ans.*

5. $a^2x^2-(\quad)+25$.
6. $(\quad)-6\,pq+4\,q^2$.
7. $(\quad)-2\,xy+y^2$.
8. $9\,r^2-12\,r+(\quad)$.
9. $16\,x^2-(\quad)+4$.
10. $(\quad)-28\,s+4\,s^2$.

11. $x^2+(\quad)+\frac{1}{4}$.

12. $100\,x^2+(\quad)+4$.

13. $9\,c^2-6\,c+(\quad)$.

14. $4\,b^2+(\quad)+9\,c^2$.

15. $m^2+(\quad)+121$.

16. $(a+b)^2+2(a+b)+(\quad)$.

17. $4\,s^2-(\quad)+\frac{1}{9}$.

18. $36\,t^2-(\quad)+36$.

19. $9\,s^2-(\quad)+81$.

20. $a^2-10\,a+(\quad)$.

21. $4\,x^2-(\quad)+25\,z^2$.

22. $\frac{1}{9}\,z^2+(\quad)+25\,t^2$.

64. Factoring a Trinomial Square. We have seen in § 63 how we may tell a trinomial square when we see it and tell also whether it is the square of the sum or of the difference of two numbers. We may easily go a step farther and tell just what binomial it is the square of; that is, we may find its two equal binomial factors. We will now illustrate this by examples.

EXAMPLE 1. Find the factors of $x^2+10\,x+25$.

SOLUTION. The square roots of x^2 and 25 are x and 5 respectively, while the sign before the remaining term, $10\,x$, is $+$. Hence we may write

$$x^2+10\,x+25=(x+5)(x+5)\quad \textit{Ans.}$$

EXAMPLE 2. Find the factors of $4\,x^2-16\,x+16$.

SOLUTION. The square roots of $4\,x^2$ and 16 are $2\,x$ and 4 respectively, while the sign before the remaining term, $16\,x$, is $-$. Hence we may write

$$4\,x^2-16\,x+16=(2\,x-4)(2\,x-4)\quad \textit{Ans.}$$

From these examples we obtain the following general rule.

RULE FOR FACTORING A TRINOMIAL SQUARE. *Find the square roots of the square terms and connect them by the sign of the remaining term.*

NOTE. Finding the two equal factors of a trinomial square in this way is the same as finding its *square root*. Thus, the answer to Example 1 above means that $x+5$ is the square root of $x^2+10\,x+25$. See § 14.

WRITTEN EXERCISES

In the following list, give the factors of those trinomials that are perfect squares; also state what is the square root of each. (See Note in § 64.)

1. x^2-2x+1.
2. x^2+2x+1.
3. x^2+4x+4.
4. x^2-4x+4
5. a^2+6a+9.
6. $a^2-8a+16$.
7. $a^2+10a+25$.
8. $r^2-12r+36$.
9. $r^2+12r+36$.
10. $9-6a+a^2$.
11. $4s^2+4s+1$.
12. $81-72s+16s^2$.
13. $100-20y+y^2$.
14. $81x^2+18x+1$.
15. $144r^2+72r+9$.
16. $144r^2-72r+1$.
17. $100a^2+84a+25$.
18. $121s^2-88s+16$.
19. $225x^2y^2-30xy+4$.
20. $16-8x+x^2$.
21. $z^2+32zy+256y^2$.
22. $49k^2+28kp+4p^2$.
23. $25m^2-20mn+4n^2$.
24. $4x^2-2xy-\frac{1}{4}y^2$.

For further exercises on this topic, see the review exercises, p. 114, *and Appendix, p.* 300.

65. The Product of the Sum and Difference of Two Numbers. Suppose we have any two numbers, a and b, and let us form their sum, $a+b$, and their difference, $a-b$. The product of this sum and difference, when multiplied out as in § 53, looks as follows:

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2+0\ -b^2 \end{array}$$

or simply

$$a^2-b^2.$$

Thus we have the following formula.

Formula VII. $\qquad (a+b)(a-b)=a^2-b^2.$

Expressed in words, this says that *the sum of any two numbers multiplied by their difference equals the difference of their squares.*

The examples below illustrate this:

EXAMPLE 1. $(x+8)(x-8)=x^2-8^2=x^2-64.$ *Ans.*

EXAMPLE 2. $(2\,xy+6)(2\,xy-6)=(2\,xy)^2-6^2$
$=4\,x^2y^2-36.$ *Ans.*

EXAMPLE 3. $(9\,x+3\,y)(9\,x-3\,y)=(9\,x)^2-(3\,y)^2$
$=81\,x^2-9\,y^2.$ *Ans.*

NOTE. Formula VII is a special case of Formula IV, for if in that formula we use a for x, $+b$ for m, and $-b$ for n, we obtain

$$(a+b)(a-b)=a^2+(-b+b)a+(-b\cdot b)=a^2-0\cdot a-b^2=a^2-b^2.$$

ORAL EXERCISES

Perform each of the following multiplications by inspection.

1. $(a+2)(a-2)$.
2. $(s+3)(s-3)$.
3. $(x-4)(x+4)$.
4. $(r-5)(r+5)$.
5. $(k-10)(k+10)$.
6. $(x+9)(x-9)$.
7. $(s-7)(s+7)$.
8. $(z+21)(z-21)$.
9. $(z-30)(z+30)$.
10. $(6+a)(6-a)$.
11. $(3+b)(3-b)$.
12. $(2\,a+5)(2\,a-5)$.
13. $(5\,r+8)(5\,r-8)$.
14. $(6\,a+3\,b)(6\,a-3\,b)$.
15. $(8\,M+5\,K)(8\,M-5\,K)$.
16. $(11\,R+4\,c)(11\,R-4\,c)$.
17. $(x^2-4)(x^2+4)$.
18. $(r^4+2)(r^4-2)$.
19. $(3\,a^3-5)(3\,a^3+5)$.
20. $(1-3\,x^2)(1+3\,x^2)$.
21. $(xy-12)(xy+12)$.
22. $(xy+rs)(xy-rs)$.
23. $(2-13\,ay)(2+13\,ay)$.
24. $(a+\frac{1}{2})(a-\frac{1}{2})$.
25. $(x^2-\frac{2}{3})(x^2+\frac{2}{3})$.
26. $\left(\frac{a}{2}+\frac{b}{3}\right)\left(\frac{a}{2}-\frac{b}{3}\right)$.
27. $(\frac{3}{4}\,r+\frac{2}{3}\,z)(\frac{3}{4}\,r-\frac{2}{3}\,z)$.
28. $(.2\,a-.3\,b)(.2\,a+.3\,b)$.
29. $(3\,x^2yz-8)(3\,x^2yz+8)$.
30. $(a^3+b^3)(a^3-b^3)$.

31. Find by Formula VII the value of $32 \cdot 28$.

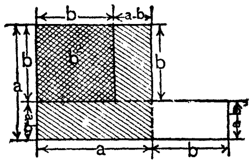
SOLUTION. $32 \cdot 28 = (30+2)(30-2) = 30^2 - 2^2 = 900 - 4 = 896$. *Ans.*

32. Find by Formula VII the value of the following (see Ex. 31).

(*a*) $22 \cdot 18$	(*d*) $29 \cdot 31$	(*g*) $49 \cdot 51$	(*j*) $34 \cdot 26$
(*b*) $62 \cdot 58$	(*e*) $37 \cdot 43$	(*h*) $41 \cdot 39$	(*k*) $61 \cdot 59$
(*c*) $47 \cdot 53$	(*f*) $88 \cdot 92$	(*i*) $72 \cdot 68$	(*l*) $123 \cdot 117$

For further exercises on this topic, see the review exercises, p. 114, *and Appendix, p.* 300.

66. Geometric Meaning of Formula VII. A careful examination of the figure here given will show that it gives the geometric meaning of Formula VII. Thus, $(a+b)(a-b)$ appears as the area of the rectangle lying along the bottom and extending the whole width of the figure, and this rectangle has the same area as that of the whole square, or a^2, less the area of the small square, or b^2.



$(a+b)(a-b) = a^2 - b^2$.

FIG. 29.

67. Factoring the Difference of Two Squares. Formula VII gives a quick method of factoring the difference of two squares, as illustrated below.

EXAMPLE 1. Factor $4x^2 - 25y^2$.

SOLUTION. $4x^2 - 25y^2 = (2x)^2 - (5y)^2$
$= (2x+5y)(2x-5y)$. *Ans.*

EXAMPLE 2. Factor $64m^2n^2 - 9p^2q^2$.

SOLUTION. $64m^2n^2 - 9p^2q^2 = (8mn)^2 - (3pq)^2$
$= (8mn+3pq)(8mn-3pq)$. *Ans.*

EXAMPLE 3. Factor $16x^4 - 1$.

SOLUTION. $16x^4 - 1 = (4x^2)^2 - 1 = (4x^2+1)(4x^2-1)$
$= (4x^2+1)(2x+1)(2x-1)$. *Ans.*

ORAL EXERCISES

Factor each of the following expressions by inspection.

1. x^2-25.
2. s^2-36.
3. $36-a^2$.
4. m^2-4.
5. b^2-49.
6. d^2-81.
7. r^2-100.
8. $1-r^2$.
9. $81-z^2$.
10. $100-b^2$.
11. a^2b^2-25.
12. $a^2-b^2c^2$.
13. $16\,x^2-25\,y^2$.
14. $25\,y^2-16\,x^2$.
15. $1-121\,c^2$.
16. $64\,x^2-25\,y^2$.
17. $121\,c^2-1$.
18. $144\,a^2-4$.
19. $16\,x^2-121\,z^2$.
20. $81\,R^2-36\,s^2$.
21. $x^2z^2-y^2$.
22. $(a+b)^2-c^2$.
23. $x^2-(y+z)^2$.
24. $(x+3\,y)^2-9\,y^2$.
25. $\frac{1}{4}\,a^2-25$.
26. $\frac{4}{9}\,r^2-\frac{1}{25}$.
27. $x^2-\frac{4}{9}\,r^2$.
28. $\dfrac{1}{x^2}-\dfrac{1}{y^2}$.
29. $\dfrac{a^2}{b^2}-\dfrac{x^2}{y^2}$.
30. $\dfrac{36}{a^2}-\dfrac{25}{b^2}$.
31. $(2\,a+b)^2-(a-b)^2$.
32. $(5\,b-4\,c)^2-(3\,a-2c)^2$.
33. $(x^3+x^2)^2-(2\,x+2)^2$.

For further exercises on this topic, see the review exercises, *p.* 114, *and Appendix, p.* 300.

EXERCISES — APPLIED PROBLEMS

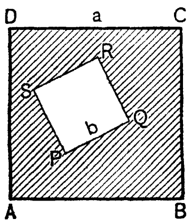


FIG. 30.

1. The figure represents a large square, whose side = a, and within it lies (in any manner) a smaller square whose side = b. Show that the *area* between the two (shaded in the figure) is represented by the expression $(a+b)(a-b)$.

[HINT. The shaded area = a^2-b^2.]

2. The result in Ex. 1 gives a quick way of determining the area between any two squares. State what it is.

3. By means of Exs. 1 and 2 answer the following question: What is the area of pavement in the street surrounding a city block one half a mile on a side, the street being 4 rods wide?

[HINT. 1 mile = 320 rods. Express your answer in square rods.]

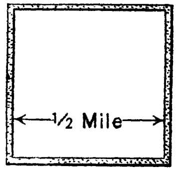


FIG. 31.

4. Show that the area between a circle of radius R and a smaller circle of radius r lying (in any manner) within it is equal to $\pi(R+r)(R-r)$.

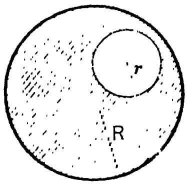


FIG. 32.

[HINT. See Ex. 25, p. 22.]

5. The diameter of a certain sunflower was 10 inches, and the black center containing the seeds (called the disk) measured 4 inches across. What was the area of the outside yellow part (called the rays)? Neglect the slight overlapping of the rays and work by Ex. 4.

6. The figure shows a right triangle whose sides are a and b and whose hypotenuse (long side) is h. Show that $a^2=(h+b)(h-b)$; also that $b^2=(h+a)(h-a)$.

[HINT. See Ex. 20, p. 21.]

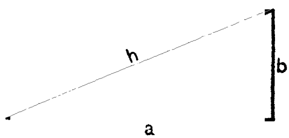


FIG. 33.

7. By means of the result in Ex. 6 answer the following: A 20-foot ladder rests against a building, the bottom of the ladder being 12 feet from the cellar wall. How far is the top from the ground?

8. Formula VII is frequently used to find the square of a number quickly by mental arithmetic. Suppose, for example, we wish to know the value of 16^2. We first take 6 away from 16, giving 10, then we add 6 to the 16, giving 22. We multiply the 10 and the 22 (as is done easily mentally) giving us 220. Now, all we have to do is to *add* 6^2, or 36, to the 220 to get the value desired. This gives 256 as the answer. The reason for this appears from the following. By Formula VII, we have

$$(16-6)(16+6)=16^2-6^2.$$

Hence, $$(16-6)(16+6)+6^2=16^2.$$

Find (mentally) in this way the values of the following:

(*a*) 15^2.

[HINT. Subtract 5 from the 15, getting 10; then add 5 to it, getting 20. Multiply these and add 5^2.]

(*b*) 14^2. [HINT. Do with 4 here as you did with 5 in (*a*).

(*c*) 12^2. (*f*) 17^2. (*h*) 22^2.

(*d*) 13^2. (*g*) 21^2. (*i*) 25^2.

(*e*) 11^2. HINT. Subtract 1 and add 1. (*j*) 31^2.

68. Further Study of the Equation. The following principle is useful, as we shall show presently, in the solving of equations.

PRINCIPLE. *If either of two numbers is equal to zero, the product of the numbers is itself zero.* For example, $2\times 0=0$; $(-4)\times 0=0$, etc.

The way in which this principle may be used in solving equations is illustrated below.

EXAMPLE 1. Solve the equation $x^2-8x+15=0$.

SOLUTION. Since the trinomial $x^2-8x+15$ may be factored (see § 58) into $(x-3)(x-5)$, the given equation may be written in the form

$$(x-3)(x-5)=0.$$

Thus, we are to have the product of two numbers equal to zero, and it will be, according to the Principle above, whenever either of the factors is equal to zero. That is, when either $x-3=0$, or $x-5=0$. But these last two equations give us $x=3$ and $x=5$. Therefore, either $x=3$ or $x=5$ is a solution (root) of our equation. *Ans.*

CHECK. When $x=3$, the left side of the given equation becomes $3^2-8\cdot 3+15$, which reduces to $9-24+15$, and this reduces (as the equation demands) to 0.

When $x=5$, the left side of the given equation becomes

$$5^2-8\cdot 5+15=25-40+15=0.$$

EXAMPLE 2. Solve the equation $x^2-4x-21=0$.

SOLUTION. Factoring, we have $(x-7)(x+3)=0$.

Whence, by the Principle above, we have either $x-7=0$, or $x+3=0$.

Therefore, the roots are 7 and -3. *Ans.*

CHECK.

$$7^2-4\cdot 7-21=49-28-21=0;\ (-3)^2-4(-3)-21=9+12-21=0.$$

NOTE. Each of the equations just considered has *two* roots. This is true of every equation which contains the second (but no higher) power of the unknown letter. Such equations are called *quadratic equations* and will be more fully considered in Chapter XVI.

WRITTEN EXERCISES

Solve the following equations by factoring. Check the first ten.

1. $x^2-7x+10=0$.
2. $x^2-5x+6=0$.
3. $x^2+8x+15=0$.
4. $x^2+7x-30=0$.

5. $x^2+4x=0.$

6. $x^2-5x=0.$

7. $x^2+5x-6=0.$

8. $x^2-x-6=0.$

9. $n^2-2n-35=0.$

10. $m^2+8m+15=0.$

11. $r^2=6+r.$

[HINT. Transpose all terms to left side.]

12. $n^2-32=-4n.$

13. $5x^2-4x=0.$

14. $3+5x=2x^2.$

15. $a^2-25=0.$

16. $r^2-36=0.$

17. $s^2-s-30=0.$

18. $x^2-x-56=0.$

19. $z^2-3z-28=0.$

20. $x^2-6x-40=0.$

21. $c^2+9c+20=0.$

22. $x^2-5x-14=0.$

23. $4x^2-8x+4=0.$

[HINT. Factor by Formula VI.]

24. $9x^2-12x+4=0.$

25. $\frac{1}{4}x^2-x+1=0.$

26. $\frac{1}{9}x^2-\frac{4}{15}x+\frac{4}{25}=0.$

27. $\frac{2}{3}r^2+\frac{1}{3}r=0.$

28. $x^2-\frac{3}{2}x+\frac{1}{2}=0.$

For further exercises on this topic, see the review exercises below, and Appendix, p. 301.

EXERCISES — REVIEW OF CHAPTER VI

1. Using Formulas I and II (§§ 48 and 50) state (orally) a different form for each of the following expressions.

(*a*) $2x^2x^2.$

(*b*) $(2xy)^2.$

(*c*) $ma^2a^3+n(ab)^3.$

(*d*) $4(abc)^2ab-5(xyz)^3x^2y^3.$

(*e*) $4x(-xy)^2-5y(-yz)^3+6z(xy)^2.$

(*f*) $(3ab)^2(xy)^3+4(2cd)^2(yz)^2+3x^2x(zw)^4.$

(*g*) $(ab)^2(ab)^3+(cd)^2(cd)^3+x^2x(x\cdot x)^4.$

(*h*) $(x^2y^2)^2-5(abcd)^3a^2b^2+4(p^2q^2r^2s^2)^2.$

[HINT. $(x^2)^2=x^2\cdot x^2=x^4.$]

(*i*) $(x^2x^3y^2z^3)^3.$

(*j*) $(-3h^2k^3l)^3-(-2g^2h^2)^2g^2h^2.$

2. Using Formula III (§ 51) state (orally) a different form for each of the following expressions.

(*a*) $3(x+y)$.
(*b*) $m(a+b)$.
(*c*) $mn(a-b)$.
(*d*) $m^2n^2(a^2+b^2+c^2)$.
(*e*) $ab(c+d)-pq(r+s)$.
(*f*) $xy(x+y)+yz(z^2+y)$.

3. Factor each of the following expressions. (See §§ 52 and 54.)

(*a*) $5\,x^4y-10\,x^3y^2$.
(*b*) $5\,m^4-10\,m^3-5\,m^2$.
(*c*) $3\,q^5-12\,q^3r^2+6\,qr^4$.
(*d*) $3\,x^3y^3+3\,x^2y^2+12\,xy$.
(*e*) $60\,m^2n^3r^2-45\,m^3n^2r^3+90\,m^4n^3r^2$.
(*f*) $12\,a^2b^2c^3-16\,a^2b^2c^2-20\,a^3b^3c^3$.
(*g*) $aq-ar+qx-rx$.
(*h*) $gh-gx-rh+rx$.
(*i*) $x^2-xy-5\,x+5\,y$.
(*j*) $x^2+xy-ax-ay$.
(*k*) $1-m+n-mn$.
(*l*) $ax-xy-ab+by$.
(*m*) $m^3+m^2-3\,m-3$.
(*n*) $3\,a^2b-9\,ab^2+qa-3\,qb$.
(*o*) $16\,ax+12\,ay-8\,bx-6\,by$.
(*p*) $mn+m-3\,n^2-3\,n-4\,n-4$.
(*q*) $px-qx-x-py+qy+y$.
(*r*) $m^2+mn+mn+n^2+m+n$.

4. By use of Formula IV (§ 57) state (orally) the result of each of the following multiplications.

(*a*) $(n-8)(n-12)$.
(*b*) $(p^2-5)(p^2-3)$.
(*c*) $(x^3-7)(x^3+6)$.
(*d*) $(a^n-5)(a^n+4)$.
(*e*) $(x^n-a)(x^n-b)$.
(*f*) $(m-2\,a)(m+3\,b)$.
(*g*) $(2\,x+1)(2\,x-3)$.
(*h*) $(2\,x-7)(2\,x+6)$.
(*i*) $(3\,y-2)(3\,y+3)$.
(*j*) $(4\,x^2+1)(4x^2-3)$.
(*k*) $(ab-3)(ab+4)$.
(*l*) $(m^2n^2-a)(m^2n^2-2\,a)$.
(*m*) $(2\,xy+y^2)(y^2-xy)$.
(*n*) $(abc+d)(abc+1)$.

5. Factor the following expressions. (See § 58.)

(*a*) $x^2+7\,x+12$.
(*b*) $x^2-7\,x+12$.
(*c*) $m^2+8\,m+12$.
(*d*) a^2+a-12.
(*e*) $b^2-5\,b-14$.
(*f*) $x^2-2\,x-15$.
(*g*) x^2-x-30.
(*h*) $y^2-6\,ay+5\,a^2$.
(*i*) $x^2-3\,nx-28\,n^2$.
(*j*) $x^2+18\,bx+80\,b^2$.
(*k*) $x^2+15\,ax+56\,a^2$.
(*l*) $x^2-11\,ax+30\,a^2$.
(*m*) $-52+9\,b+b^2$.
(*n*) $100-25\,x+x^2$.

6. By use of Formulas V and VI (§§ 59 and 60) expand each of the following expressions by inspection.

(*a*) $(2a+x)^2$. (*b*) $(ab+cd)^2$. (*c*) $(4x+3y)^2$. (*d*) $(4x-3y)^2$. (*e*) $(xy+x^2y)^2$. (*f*) $(3xy-2x^2y)^2$. (*g*) $(x^3+y^3)^2$. (*h*) $(a^5-b^5)^2$. (*i*) $(x^n-y^n)^2$. (*j*) $(1+3a^2b)^2$. (*k*) $(3a^2+5b^3)^2$. (*l*) $(ma^m-nb^n)^2$.

7. Show that each of the following trinomials is a perfect square and find its square root. (See §§ 63 and 64.)

(*a*) $x^2-8x+16$. (*b*) $a^2-10a+25$. (*c*) $1+4y+4y^2$. (*d*) $1-6b+9b^2$. (*e*) $4x^2-4x+1$. (*f*) $9m^2-30m+25$. (*g*) $4x^2y^2-16xy+16$. (*h*) $4m^2n^2-4amn+a^2$. (*i*) $a^2b^2c^2-2abcde+d^2e^2$. (*j*) $9x^2y^2-12xyz+4z^2$. (*k*) $9a^2b^2c^2+12abc+4$. (*l*) $m^2a^4b^4-2ma^2b^2n^2cd+n^4c^2d^2$.

8. By use of Formula VII (§ 65) state the result of each of the following multiplications.

(*a*) $(x-1)(x+1)$. (*b*) $(x^2+1)(x^2-1)$. (*c*) $(x^2+y^2)(x^2-y^2)$. (*d*) $(2x+3y)(2x-3y)$. (*e*) $(2ab+3)(2ab-3)$. (*f*) $(xy+z^2)(xy-z^2)$. (*g*) $(3m^2n-q)(3m^2n+q)$. (*h*) $(3x^3-4y^2)(3x^3+4y^2)$. (*i*) $(-5x-y)(-5x+y)$. (*j*) $(-3a^3b+c^2d^2)(-3a^3b-c^2d^2)$. (*k*) $(x^{m-1}+y^{n+1})(x^{m-1}-y^{n+1})$. (*l*) $[(a+b)^2+3][(a+b)^2-3]$. (*m*) $[2x-1-y][2x-1+y]$.

9. Factor each of the following expressions. (See § 67.)

(*a*) a^2-16. (*b*) $25a^2-36b^2$. (*c*) $400x^2-81y^2$. (*d*) x^2y^2-256. (*e*) $x^{2n}-y^{2m}$. (*f*) $x^2-(y+z)^2$. (*g*) $9m^2-(p+q)^2$. (*h*) $x^2-(2x-3y)^2$. (*i*) $81x^2-(3m-2n)^2$. (*j*) $(2a+3b)^2-(a+b)^2$. (*k*) $(r-2s)^2-(r-3)^2$. (*l*) $(a+b+c)^2-(a-b-c)^2$.

10. Solve (by factoring) each of the following equations. (See § 68.)

(a) $x^2-5x+4=0$.

(b) $r^2+6r+8=0$.

(c) $x^2-3x=40$.

(d) $y^2-15y=54$.

(e) $m^2-20m=96$.

(f) $t^2+63=16t$.

(g) $v^2-60=11v$.

(h) $n^2+11n+30=0$

(i) $3s=s^2-88$.

(j) $160=x^2-6x$.

(k) $4p=192-p^2$.

(l) $600=y^2-10y$.

(m) $g^2+15g-34=0$.

(n) $(x-1)^2-3(x-1)$

EXERCISES — APPLIED PROBLEMS

1. A certain rectangle is 2 feet longer than wide, and its area is 8 square feet. What are its dimensions?

HINT. Let $x=$ the width. Then $x+2=$ the length. Now form an equation for x, transpose all its terms to the left side and solve as in § 68. Keep only the $+$ root, since the $-$ root has no meaning in this problem.

2. The perimeter (distance around) a certain rectangle is 10 feet, while the area is 6 feet. What are the dimensions?

3. The figure represents a rectangular plot of ground, within which are two equal square flower beds. The border (shaded in the figure) is everywhere 3 feet wide and contains 201 square feet. What is the length of side of each flower bed, and what are the dimensions of the rectangle?

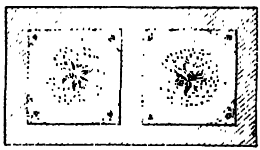


FIG. 34.

4. In the figure are two circles so placed that the inner one just *touches* the outer one. Show that the area of the crescent thus formed (shaded in the figure) may be expressed in the form $\pi(R+r)(R-r)$, where R is the radius of the large circle, and r that of the small circle.

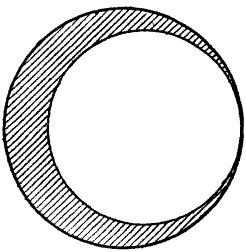


FIG. 35.

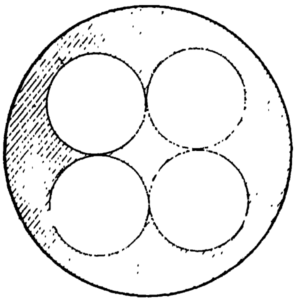


FIG. 36.

5. If, in the figure at the left, the radius of the large circle is R and the radius of each of the smaller circles is r, show that the shaded area, A, is given by the formula

$$A = \pi(R+2r)(R-2r).$$

6. Figure 37 represents a piece of iron tubing whose outer diameter is D and whose inner diameter is d. If the length is l, show that the volume V occupied by the iron is expressed by the formula

$$V = \pi l\left(\frac{D+d}{2}\right)\left(\frac{D-d}{2}\right).$$

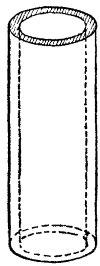


FIG. 37.

7. The result in Ex. 6 may be expressed in words as follows: "To find the volume of a circular tube, multiply π times the length by half the sum of the two diameters by ——." Finish the sentence. Does this give us a convenient rule for determining any such volume? Explain.

8. One side of a right triangle measures 12 feet, while the hypotenuse is 2 feet longer than the other side. Find the length of the other side.

For further exercises on this Chapter, see Appendix, pp. 298–301.

CHAPTER VII

DIVISION AND FACTORING †

69. Division. The process of finding one factor when the product and other factor are given is called ***division***.

The ***dividend*** is the given product, the ***divisor*** is the given factor, and the ***quotient*** is the required factor.

Thus, to divide $6\,ab$ by $3\,a$ means to find the number which when multiplied by $3\,a$ gives $6\,ab$. Here the dividend is $6\,ab$, the divisor is $3\,a$, and the answer (or quotient) is evidently $2\,b$.

70. Law of Signs for Division. In § 27 we saw that the sign of a quotient is $+$ whenever the dividend and divisor have *like* signs, and is $-$ whenever they have *unlike* signs. This may be stated in a table as follows:

$$\begin{aligned} +ab \div (+b) &= +a \\ -ab \div (+b) &= -a \\ -ab \div (-b) &= +a \\ +ab \div (-b) &= -a \end{aligned}$$

71. Law of Exponents for Division. In dividing x^5 by x^3 we may proceed, as in arithmetic, by removing equal factors from dividend and divisor, the work appearing as below.

$$\frac{x^5}{x^3} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^{(5-3)} = x^2. \quad Ans.$$

Similarly, we see that $x^6 \div x^2$ is equal to $x^{(6-2)}$, or x^4. In general, we have the following rule.

† § 27 should be read again at this point.

RULE FOR FINDING THE QUOTIENT OF TWO POWERS OF THE SAME NUMBER. *The exponent of the quotient of two powers of the same number is equal to the exponent of the dividend* ***diminished by*** *that of the divisor.* Stated in a formula, the rule becomes

Formula VIII. $$\frac{x^m}{x^n} = x^{(m-n)}.$$

When $m=n$ this formula gives $x^n/x^n = x^{(n-n)} = x^0$. But, we also have $x^n/x^n = 1$ because numerator and denominator here are the *same*. Therefore, *the meaning of x^0 must always be taken as* 1; that is,

$$x^0 = 1.$$

The pupil should carefully compare Formulas I and VIII.

ORAL EXERCISES

State the quotient for each of the following indicated divisions.

1. $5\overline{)-10}$
2. $-5\overline{)15}$
3. $-3\overline{)-15}$
4. $-2\overline{)10}$
5. $\frac{35}{-7}$.
6. $\frac{-42}{6}$.
7. $\frac{6\,ab}{2\,a}$.
8. $\frac{-4\,b}{2\,b}$.
9. $\frac{7\,abc}{ab}$.
10. $\frac{x^3}{x}$.
11. $\frac{a^9}{a^7}$.
12. $\frac{(2\,g)^3}{(2\,g)}$.
13. $\frac{(ab)^5}{(ab)^4}$.
14. $\frac{(3\,xy)^7}{(3\,xy)^3}$.
15. $\frac{(\frac{1}{2}\,p^2q)^5}{(\frac{1}{2}\,p^2q)^3}$.
16. $\frac{(abc)^3}{abc}$.
17. $\frac{(a+b)^4}{(a+b)^3}$.
18. $\frac{(x-y+z)^s}{(x-y+z)^r}$.

72. To Divide a Monomial by a Monomial.

EXAMPLE 1. Divide $35\,c^4d^2$ by $5\,c^2d$.

SOLUTION. $$5\,c^2d\overline{)35\,c^4d^2}$$
$$7\,c^2d. \quad \textit{Ans.}$$

Note that the process consists in first dividing the 35 by 5 to

obtain the new coefficient 7, then dividing the c^4 by c^2, giving c^2 (by Formula VIII), then dividing the d^2 by the d, giving d.

CHECK. $(5\,c^2d)\times(7\,c^2d)=35\,c^4d^2.$

EXAMPLE 2. Divide $-10\,x^3y^3z^4$ by $2\,x^2z$.

SOLUTION.
$$2\,x^2z\overline{)-10\,x^3y^3z^4}$$
$$-5\,xy^3z^3.$$

Here the process is the same as in Example 1, but it is to be observed that the y^3 which occurs in the dividend occurs without change in the quotient since the divisor contains no y factor.

CHECK. $(2\,x^2z)\times(-5\,xy^3z^3)=-10\,x^3y^3z^4.$

A careful examination of the two examples above enables one to work all similar examples without difficulty.

WRITTEN EXERCISES

Find the quotient for each of the following indicated divisions, and check each answer.

1. $-4\,ab\overline{)28\,a^2b^3}$
2. $3\,xy^2\overline{)9\,x^3y^3z^2}$
3. $\dfrac{-16\,x^3y^3z^3}{4\,xy^2z}.$
4. $\dfrac{27\,m^3n^2p}{-3\,m^2p}.$
5. $\dfrac{4\,a^4b^3c^5}{20\,a^2bc^3}.$

[HINT. The coefficient of the quotient is $\frac{4}{20}$, or $\frac{1}{5}$.]

6. $\dfrac{-4\,x^7y^4z^5}{32\,x^4y^2z^3}.$
7. $\dfrac{x^3y^2z}{x^3y}.$

$\left[\text{HINT. } \dfrac{x^3}{x^3}=x^0=1. \text{ See § 71.}\right]$

8. $\dfrac{\frac{4}{3}\pi r^3}{2\,\pi r}.$
9. $\dfrac{-13\,g^3h^2k}{-7\,g^2k}.$
10. $\dfrac{4(2\,a)^3(3\,b)^4}{2(2\,a)^2(3\,b)^2}.$
11. $\dfrac{3\,ab(a+b)^2}{-2(a+b)}.$
12. $\dfrac{2\,a^2(x+y)^3}{-a(x+y)}.$
13. $\dfrac{\frac{2}{3}(x+y)^2(x-y)^3}{\frac{3}{4}(x+y)(x-y)}.$
14. $\dfrac{(\frac{1}{2})^3(\frac{2}{3})^2}{(\frac{1}{2})^2(\frac{2}{3})}.$
15. $\dfrac{-15\,x^{2n-1}y^{3n}}{5\,x^{n-1}y^{2n}}.$

73. To Divide a Polynomial by a Monomial.

EXAMPLE 1. Divide $8\,x^2y-4\,x^2y^2$ by $4\,xy$.

SOLUTION.
$$4\,xy\overline{)8\,x^2y-4\,x^2y^2}$$
$$2\,x-xy. \quad \textit{Ans.}$$

Note that the process consists in dividing *each* term of the polynomial, as in § 72, keeping due account of the connecting signs.

CHECK. $4\,xy(2\,x-xy)=8\,x^2y-4\,x^2y^2$.

EXAMPLE 2. Divide $9\,a^2b^2c^2+12\,a^2b-15\,b^2cd$ by $-3\,b$.

SOLUTION.
$$-3\,b\overline{)\;9\,a^2b^2c^2+12\,a^2b-15\,b^2cd}$$
$$-3\,a^2bc^2-4\,a^2+5\,bcd. \quad \textit{Ans.}$$

CHECK. $-3\,b(-3\,a^2bc^2-4\,a^2+5\,bcd)=9\,a^2b^2c^2+12\,a^2b-15\,b^2cd$.

WRITTEN EXERCISES

Find the quotient in each of the following and check your answer.

1. $8\,a^4b^2\overline{)24\,a^6b^2+32\,a^5b^3-16\,a^4b^3}$

2. $\dfrac{6\,x^2yz+12\,xy^2z-24\,xyz^2}{-3\,xyz}$.

3. $\dfrac{m^3np+mn^3p}{mn}$.

4. $\dfrac{-a+a^2b-a^3c-a^4d+a^5e}{-a}$.

5. $\dfrac{a(b-c)^3-b(b-c)^2+c(b-c)}{(b-c)}$.

6. $(-3\,x^3+7\,x^2-x)\div x$.

7. $(-m-m^2-m^3-m^4)\div(-m)$.

8. $(x^n+2\,x^{n+1}+3\,x^{n+2}+4\,x^{n+3}+5\,x^{n+4})\div x^n$.

9. $(r^{8m}s^{6n}-3\,r^{6m}s^{8n}-5\,r^{4m}s^{10n})\div(-5\,r^{4m}s^{6n})$.

74. To Divide a Polynomial by a Polynomial.

EXAMPLE 1. Divide $3x^2+17x+20$ by $x+4$.

SOLUTION. *Dividend.*	$3x^2+17x+20$	$x+4$	*Divisor.*
3 times $(x+4)$ gives	$3x^2+12x$	$3x+5$	*Quotient.* *Ans.*
Subtracting gives	$5x+20$		
5 times $(x+4)$ gives	$5x+20$		
Subtracting gives	0		

EXPLANATION. The solution above consists of several steps taken in a definite order as follows.

(1) The terms of both dividend and divisor are arranged according to the descending powers of x. (See § 33.)

(2) The first term in the dividend is divided by the first term of the divisor, giving the *first* term in the quotient. That is, the $3x^2$ is divided by the x, giving $3x$.

(3) The $3x$ thus obtained is now multiplied by the divisor and the result subtracted from the dividend, giving $5x+20$. This $5x+20$ is therefore what is left of the dividend after the terms in it that come from multiplying the divisor by the *first* term of the quotient have been subtracted.

(4) $5x+20$ is now treated as a *new* dividend and the process carried on as before, this time to obtain the *second* term of the quotient. That is, the $5x$ is divided by the x of the divisor, giving 5 as the second term of the quotient.

(5) The 5 thus obtained is now multiplied by the divisor and subtracted to form a new dividend as before. This, however, turns out in the present instance to give zero (all terms canceling) so no further terms are left in the dividend to be considered. Hence the complete quotient is $3x+5$.

CHECK. Multiplying the divisor, $x+4$, by the quotient, $3x+5$, in the way shown in § 53, gives the dividend, $3x^2+17x+20$.

EXAMPLE 2. Divide $6x^3-7x^2+1$ by $2x-1$.

SOLUTION. Observe that the dividend contains an x^3 term and an x^2 term but *no* x term; that is, the *first* power of x is missing. In such cases it is best in arranging the work to leave a space for the

missing term, after which the process is the same as in Example 1.

The process is given in full below.

$$\begin{array}{rrrr|l}
6x^3 & -7x^2 & +(\ \ ) & +1 & \underline{2x-1} \quad \textit{Divisor.} \\
\underline{6x^3} & \underline{-3x^2} & & & 3x^2-2x-1 \quad \textit{Quotient.} \quad \textit{Ans.} \\
 & -4x^2 & +(\ \ ) & +1 & \\
 & \underline{-4x^2} & \underline{+2x} & & \\
 & & -2x & +1 & \\
 & & \underline{-2x} & \underline{+1} & \\
 & & & 0 &
\end{array}$$

In case the result of the last subtraction is *not* zero (note that it was zero in Examples 1 and 2) then, as in arithmetic, the final result of the division is a quotient and a ***remainder***. This is illustrated in the following example.

EXAMPLE 3. Divide $5x^4-x^3+x+1$ by x^2-1.

SOLUTION.

$$\begin{array}{rrrrr|l}
5x^4 & -x^3 & +(\ \ ) & +x & +1 & \underline{x^2-1} \quad \textit{Divisor.} \\
\underline{5x^4} & & \underline{-5x^2} & & & 5x^2-x+5 \quad \textit{Quotient.} \quad \textit{Ans.} \\
 & -x^3 & +5x^2 & +x & & \\
 & \underline{-x^3} & & \underline{+x} & & \\
 & & 5x^2 & & +1 & \\
 & & \underline{5x^2} & & \underline{-5} & \\
 & & & & +6 & \textit{Remainder.}
\end{array}$$

CHECK. As in arithmetic, when the remainder is added to the product of the divisor and quotient, the result should be the dividend. In other words, $(x^2-1)(5x^2-x+5)+6$ should equal $5x^4-x^3+x+1$, and we readily find this to be so. The work is left to the pupil.

WRITTEN EXERCISES

Carry out each of the following indicated divisions, and check each answer.

1. $(3x^2-2x-1)\div(x-1)$.
2. $(3x^3-4x^2+2x-1)\div(x-1)$.
3. $(15x^2+x-2)\div(3x-1)$.
4. $(4y^3+2y^2-1)\div(2y-1)$.

[HINT. The first power of y is missing in the dividend. See Example 2 in § 74.]

5. $(a^3-3a^2+2a-6)\div(a^2+2)$.

6. $(m^5+m^4-m^3-1)\div(m^2-1)$.

7. $(x^3+x^2-x+2)\div(x^2-x+1)$.

Solution. Here, the divisor is a *trinomial* (instead of a binomial as in preceding examples). The division, however, is carried out as before. Thus,

$$\begin{array}{l|ll}
x^3+x^2-x+2 & \underline{x^2-x+1} & \textit{Divisor.} \\
\underline{x^3-x^2+x} & x+2 & \textit{Quotient.}\quad \textit{Ans.} \\
2x^2-2x+2 & & \\
\underline{2x^2-2x+2} & & \\
0 & &
\end{array}$$

8. $(6y^3-5y^2+7y-2)\div(2y^2-y+2)$.

9. $(a^4+16+4a^2)\div(2a+a^2+4)$.

[Hint. First arrange in descending powers of a.]

10. $(x^5-61x-60)\div(x^2-2x-3)$.

11. $(x^4-1)\div(x+1)$.

12. $(x^5+1)\div(x+1)$.

13. $(x^3+1)\div(x^2-x+1)$.

14. $(x^4-1)\div(x^3-x^2+x-1)$.

15. Find the *quotient and remainder* in each of the following indicated divisions. (See Example 3 in § 74.)

(*a*) $(2x^2+3x+1)\div(x+2)$.

(*b*) $(6x^3-5x^2+7x+1)\div(3x-1)$.

(*c*) $(x^4-3x^3+x^2+2x-1)\div(x^2-x-2)$.

16. Divide $2x^2+xy-y^2$ by $x+y$.

[Hint. Here *two* letters, x and y, occur, but the process is the same as before. The quotient is $2x-y$.]

17. Divide $a^3-2a^2b+2ab^2-b^3$ by $a-b$.

18. Divide $m^3n+2mn^3-m^2-2n^2$ by m^2+2n^2.

[Hint. Arrange in descending powers of m.]

19. Divide x^4-y^4 by $x-y$.

20. Divide x^4+y^4 by $x+y$.

21. Divide $x^3+y^3+z^3-3xyz$ by $x+y+z$.

For further exercises on this topic, see Appendix, p. **301.**

75. * **Special Cases of Division.**† We found in § 65 that a^2-b^2 was the product of $(a-b)$ and $(a+b)$; that is, $a^2-b^2=(a-b)(a+b)$. This is the same as saying that $(a^2-b^2)\div(a-b)=a+b$, or that $(a^2-b^2)\div(a+b)=a-b$. Whence, we may say that a^2-b^2 is divisible by either $a-b$ or $a+b$ *without a remainder;* or, as we often say, the division is *exact*. This also appears by actual division as follows.

$$\begin{array}{l|l}
a^2+(\quad)-b^2 & a-b \\
\cline{2-2}
\underline{a^2-ab} & a+b \\
\quad +ab-b^2 & \\
\quad \underline{+ab-b^2} & \\
\qquad\quad 0 &
\end{array}
\qquad\qquad
\begin{array}{l|l}
a^2+(\quad)-b^2 & a+b \\
\cline{2-2}
\underline{a^2+ab} & a-b \\
\quad -ab-b^2 & \\
\quad \underline{-ab-b^2} & \\
\qquad\quad 0 &
\end{array}$$

Suppose now we consider the difference of two *cubes:* a^3-b^3. Is this likewise divisible by either $a-b$ or $a+b$ *without a remainder;* that is, is it "exactly divisible" by $a-b$ and $a+b$? To answer, let us divide out and see.

$$\begin{array}{l|l}
a^3+(\quad)+(\quad)-b^3 & a-b \\
\cline{2-2}
\underline{a^3-a^2b} & a^2+ab+b^2 \\
\quad a^2b+(\quad) & \\
\quad \underline{a^2b+ab^2} & \\
\qquad -ab^2-b^3 & \\
\qquad \underline{-ab^2-b^3} & \\
\qquad\qquad\quad 0 &
\end{array}
\qquad\qquad
\begin{array}{l|l}
a^3+(\quad)+(\quad)-b^3 & a+b \\
\cline{2-2}
\underline{a^3+a^2b} & a^2-ab+b^2 \\
\quad -a^2b+(\quad) & \\
\quad \underline{-a^2b-ab^2} & \\
\qquad\quad ab^2-b^3 & \\
\qquad\quad \underline{ab^2+b^3} & \\
\qquad\qquad -2\,b^3 & \textit{Remainder.}
\end{array}$$

Thus we see that a^3-b^3 is exactly divisible by $a-b$, the quotient being a^2+ab+b^2, and we see also that a^3-b^3 is *not* exactly divisible by $a+b$, since we then have the remainder $-2\,b^3$.

Similarly, if we consider the *sum* (instead of the difference) of two cubes; that is, if we consider $\mathbf{a^3+b^3}$, we find (see Ex. 1 below) that it is exactly divisible by $a+b$, the quotient being a^2-ab+b^2, and we find also that a^3+b^3 is *not* exactly divisible by $a-b$, the remainder being $2\,b^3$.

† This may be omitted at the discretion of the teacher from the first year course. The same is true of the other articles and exercises that are starred (*) hereafter.

These results give us the following formulas.

* **Formula** IX *a*. $\boldsymbol{a^3-b^3=(a-b)(a^2+ab+b^2)}$.

* **Formula** IX *b*. $\boldsymbol{a^3+b^3=(a+b)(a^2-ab+b^2)}$.

Note. These Formulas give us the following useful rule: *The difference of the cubes of two numbers is divisible by the difference of the numbers, while the sum of the cubes of two numbers is divisible by the sum of the numbers.*

EXERCISES

1. Show that (as stated above) the quotient of a^3+b^3 divided by $a+b$ is a^2-ab+b^2.

2. Show that a^3+b^3 is *not* exactly divisible by $a-b$, but that there is a remainder of $2\,b^3$.

3. Show that a^4-b^4 is exactly divisible by either $a+b$ or $a-b$.

4. Show that a^2+b^2 is *not* exactly divisible by either $a-b$ or $a+b$. Find the remainder in each case.

5. Show that a^4+b^4 is not exactly divisible by either $a-b$ or $a+b$.

76. * **Factoring the Sum or Difference of Two Cubes.** Formulas IX *a* and IX *b* show that the sum of two cubes, or the difference of two cubes, may always be expressed as the product of two factors.

Example 1. Factor $8\,x^3-27\,y^3$.

Solution. $8\,x^3-27\,y^3=(2\,x)^3-(3\,y)^3$

$=(2\,x-3\,y)[(2\,x)^2+(2\,x)(3\,y)+(3\,y)^2]$ (Formula IX *a*)

$=(2\,x-3\,y)(4\,x^2+6\,xy+9\,y^2)$. *Ans.*

Example 2. Factor $m^3n^3+64\,p^3$.

Solution. $m^3n^3+64\,p^3=(mn)^3+(4\,p)^3$

$=(mn+4\,p)[(mn)^2-(mn)(4\,p)+(4\,p)^2]$. (Formula IX *b*)

$=(mn+4\,p)(m^2n^2-4\,mnp+16\,p^2)$. *Ans.*

EXERCISES

Factor each of the following expressions.

1. $a^3-8\,b^3$.

2. $a^3+8\,b^3$.

3. x^3-1.

4. x^3+1.

5. $8\,x^3+125$.

6. $1+m^3$.

7. $a^3b^3-c^3d^3$.

8. $(x-y)^3+(x+y)^3$.

9. $8(m+n)^3+125\,n^3$.

10. $1-(a-b)^3$.

11. $(x-y)^3-8$.

12. $x^3y^6z^9-216$.

77. Further Study of Factoring. In Chapter VI we saw how to factor certain expressions. Thus, $4x^2-9y^2$ may be factored by Formula VII into $(2x+3y)(2x-3y)$. Likewise, such an expression as $ac+bc+3a+3b$ can be factored by first writing it in the form $c(a+b)+3(a+b)$ and then noting that this is the same as $(c+3)(a+b)$, all of which employs Formula III.

These are examples in which a *single* formula suffices to get the answer, but we often have examples in which two or more formulas are needed at the same time. Thus, in factoring a^3b-ab^3 we first employ Formula III to take out the factor ab. This gives $a^3b-ab^3=ab(a^2-b^2)$. But a^2-b^2 is itself factorable into $(a+b)(a-b)$ by Formula VII. Therefore, the *final* answer here is $ab(a+b)(a-b)$. Other illustrations of this idea follow immediately below. Note that the *final* answer in every case contains no factors which can *themselves* be still further broken up into other factors.

EXAMPLE 1. Factor x^2-y^2+x+y.

SOLUTION.
$$\begin{aligned} x^2-y^2+x+y &= (x^2-y^2)+(x+y) \\ &= (x+y)(x-y)+(x+y) && \text{(Formula VII)} \\ &= (x+y)(x-y+1). \quad \textit{Ans.} && \text{(Formula III)} \end{aligned}$$

EXAMPLE 2. Factor $4(a^2-b^2)-3(a+b)$.

SOLUTION. $4(a^2-b^2)-3(a+b)$
$$\begin{aligned} &= 4(a-b)(a+b)-3(a+b) && \text{(Formula VII)} \\ &= (a+b)[4(a-b)-3] && \text{(Formula III)} \\ &= (a+b)(4a-4b-3). \quad \textit{Ans.} \end{aligned}$$

EXAMPLE 3. Factor $4a^2-9b^2+4a-6b$.

SOLUTION.
$$\begin{aligned} 4a^2-9b^2+4a-6b &= (4a^2-9b^2)+2(2a-3b) \\ &= (2a+3b)(2a-3b)+2(2a-3b) \\ &= (2a-3b)(2a+3b+2). \quad \textit{Ans.} \end{aligned}$$

NOTE. A common error is to think that the factoring of one or more *parts* of an expression is equivalent to factoring the expression itself. Thus, in the Example 3 just considered, where we had

to factor $4\,a^2-9\,b^2+4\,a-6\,b$, we can of course factor the part $4\,a^2-9\,b^2$, giving $(2\,a+3\,b)\,(2\,a-3\,b)$, but this does *not* give us the factors of the *given* (whole) expression, $4\,a^2-9\,b^2+4\,a-6\,b$. Observe that any expression may be said to have been factored only when it has *all* been put into the form of a product of two or more factors.

78. Summary of Factoring. All the examples in factoring which we have thus far considered have been worked by use of the following formulas:

I.	$x^m x^n = x^{m+n}.$	(§ 48)
II.	$(xy)^m = x^m y^m.$	(§ 50)
III.	$ab+ac=a(b+c).$	(§ 51)
IV.	$x^2+(m+n)x+mn=(x+m)(x+n).$	(§ 57)
V.	$a^2+2\,ab+b^2=(a+b)(a+b).$	(§ 59)
VI.	$a^2-2\,ab+b^2=(a-b)(a-b).$	(§ 60)
VII.	$a^2-b^2=(a+b)(a-b).$	(§ 65)
VIII.	$\dfrac{x^m}{x^n}=x^{m-n}.$	(§ 71)
* IX *a*.	$a^3-b^3=(a-b)(a^2+ab+b^2).$	(§ 75)
* IX *b*.	$a^3+b^3=(a+b)(a^2-ab+b^2).$	(§ 75)

EXERCISES

Each expression in the following list may be factored by the formulas in § 78. Either a single one of the formulas is necessary, or several of them in the manner shown in § 77. Before attempting these, the pupil will find it desirable to review the exercises in factoring in Chapter VI and read § 77 carefully.

Factor each of the following expressions.

1. $x^2-ax+cx-ac.$

[HINT. Write as $x(x-a)+c(x-a)$.]

2. $y^3+y^2+y+1.$

[HINT. Write as $y^2(y+1)+y+1$.]

3. $z^3-z^2-z+1.$

[HINT. Write as $(z^3-z^2)-(z-1)$.]

4. $2x^3-8x^2y+8xy^2$.

[HINT. Write as $2x(x^2-4xy+4y^2)$ and apply Formula VI, § 78.]

5. $x^2-11x+30$.

[HINT. This comes under Formula IV, § 78. See § 58.]

6. $x^2+3ax-3a-x$.

7. $a^4+3a^2b^2-4b^4$.

8. $8x-8x^3y^4$.

9. x^2+y^2-2xy.

[HINT. Rearrange the terms.]

10. a^3+2a^2+4a+8.

11. x^4-13x^2+36.

12. $x^4+y^4-2x^2y^2$.

13. $x^4-(x-2)^2$.

[HINT. The answer contains *three* factors.]

14. $1-a^2-b^2-2ab$.

[HINT. Write in the form $1-(a^2+b^2+2ab)$.]

15. $a^2b^2c^2-4b^2c^2$.

16. $m^2+4mn+4n^2-16$.

17. $2xy-x^2-y^2+1$.

18. $9a^2-6a^3+a^4$.

19. $a^2n^2+a^2m^2-b^2m^2-b^2n^2$.

20. $ax-b-a+bx$.

21. $ab^2-2abc+ac^2$.

22. $1+9c^2+6c$.

23. $(x^2-1)^2+(2x+3)(x-1)^2$.

24. $a^2-b^4-a^2x^2+b^4x^2$.

25. $3x^3-3x+4x^4-4x^2$.

26. $a^4-4b^4+a^2+2b^2$.

27. $1-4a^2b^2c^2-9x^2y^2z^2+12abcxyz$.

28. $xy-1+x-y$.

29. $a^2+(b-2bx^2)ay-2b^2x^2y^2$.

30. $1-a^2b^2-x^2y^2+2abxy$.

31. $(a+b)^2(x-y)-(a+b)(x^2-y^2)$.

32. $(x^2-y^2)^2-(x^2-xy)^2$.

33. $m^3+n^2-mn-mn^2$.

***34.** $x^3+y^3+x^2-y^2$.

***35.** $(x+1)^3-x^6$.

***36.** $5x^7y-5xy^4$.

***37.** $x^3-27-7(x-3)$.

38. $(1-2x)^2-x^4$.

39. $x^3y-10x^2y^2z^2+25xy^3z^4$.

40. x^4-18x^2+81.

***41.** $1+(x+1)^3$.

***42.** $x^3+15x^2+75x+125$.

***43.** $3ab(a+b)+a^3+b^3$.

LAPLACE

(*Pierre Simon Laplace*, 1749–1827)

Famous in mathematics for his researches, which were of a most advanced kind, and especially famous in astronomy for his enunciation of the Nebular Hypothesis. Interested also in physics and at various times held high political offices under Napoleon.

CHAPTER VIII

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

PART I. HIGHEST COMMON FACTOR

79. Common Factors. In arithmetic a factor of each of two or more numbers is called a ***common factor*** of the numbers. Thus,

3 is a common factor of 9 and 15;
5 is a common factor of 10, 15, and 25.

In the same way, we say in algebra that

x is a common factor of $2x$ and $5x$;
y is a common factor of $3y$ and y^2;
rs is a common factor of $2r^2s^2$ and rs;
$a+b$ is a common factor of $(a+b)^2$ and $a(a+b)$.

80. Prime Factors. A number that has no factor except itself and unity is called in arithmetic a prime number. Such a number when used as a factor is called a ***prime factor.***

Thus,

the prime factors of 10 are 5 and 2;
the prime factors of 36 are 3, 3, 2, and 2.

In the same way, we say in algebra that

the prime factors of $3abc$ are 3, a, b, and c;
the prime factors of $4x^2y$ are 2, 2, x, x, and y;
the prime factors of $a^2b(a^2-b^2)$ are a, a, b, $a-b$, and $a+b$.

81. To Find Common Factors. As soon as we factor each of several numbers into their prime factors, we can easily pick

out the common factors. For example, to find the common factors of 54, 90, 108, and 180, we have

$$54 = 3 \cdot 3 \cdot 2 \cdot 3,$$
$$90 = 3 \cdot 3 \cdot 2 \cdot 5,$$
$$108 = 3 \cdot 3 \cdot 2 \cdot 3 \cdot 2,$$
$$180 = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 5.$$

The common factors are therefore 3, 3, and 2.

The same process is followed in algebra. Thus, in finding the common factors of abc, a^2b, ab^2, and $3\,ab$, we write

$$abc = a \cdot b \cdot c,$$
$$a^2b = a \cdot a \cdot b,$$
$$ab^2 = a \cdot b \cdot b,$$
$$3\,ab = 3 \cdot a \cdot b.$$

The common factors are therefore a and b.

82. Highest Common Factor (H. C. F.). The product of all the common prime factors of two or more numbers or expressions is called their ***highest common factor***. It is called the *highest* common factor because it contains *all* the common factors. The abbreviation for it is ***H. C. F.***

For example, to find the H. C. F. of 90, 60, and 120, we write

$$90 = 3 \cdot 2 \cdot 5 \cdot 3,$$
$$60 = 3 \cdot 2 \cdot 5 \cdot 2,$$
$$120 = 3 \cdot 2 \cdot 5 \cdot 2 \cdot 2.$$

Since 3, 2, and 5 are the only common factors, their product, 30, is the H. C. F.

We find the H. C. F. of algebraic expressions in the same way, as is illustrated in the following examples.

EXAMPLE 1. Find the H. C. F. of $6\,a^3b^3$, $2\,ab^2$, and $8\,a^3b^2$.

SOLUTION.
$$6\,a^3b^3 = 3 \cdot 2 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b,$$
$$2\,ab^2 = 2 \cdot a \cdot b \cdot b,$$
$$8\,a^3b^2 = 2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot a \cdot b \cdot b.$$

Picking out all the factors common to the three expressions, we see

that they are 2, a, b, and b. Hence the H. C. F. is $2 \cdot a \cdot b \cdot b$, or $2\,ab^2$. *Ans.*

EXAMPLE 2. Find the H. C. F. of $a^2-4\,a+4$, $a^2-2\,a$, and $a^2-6\,a+8$.

SOLUTION.
$$a^2-4\,a+4=(a-2)(a-2), \qquad \text{(Formula VI)}$$
$$a^2-2\,a=a(a-2),$$
$$a^2-6\,a+8=(a-2)(a-4). \qquad \text{(Formula IV)}$$
The only common factor being $a-2$, the H. C. F. is $a-2$. *Ans.*

EXAMPLE 3. Find the H. C. F. of

$$3\,x^2+3\,x-18,\ 6\,x^2+36\,x+54, \text{ and } 9\,x^2-81.$$

SOLUTION.
$$3\,x^2+3\,x-18=3(x^2+x-6)=3(x+3)(x-2),$$
$$6\,x^2+36\,x+54=6(x^2+6\,x+9)=2\cdot 3(x+3)(x+3),$$
$$9\,x^2-81=9(x^2-9)=3\cdot 3(x+3)(x-3).$$
The common factors being 3 and $(x+3)$, the H. C. F. is $3(x+3)$, or $3\,x+9$. *Ans.*

83. Important Property of the H. C. F. Since the H. C. F. of several expressions is always made up of the factors common to them all, it is an exact divisor of each of the expressions.

Thus, the H. C. F. of $5\,a^2b^3$, $3\,a^2b^2(c+d)$, $4\,a^3b^3$ is a^2b^2. Observe that this is contained in the first expression $5\,b$ times, in the second expression $3(c+d)$ times, and in the third expression $4\,ab$ times. For this reason, the highest common factor is called in arithmetic *the greatest common divisor*, and is represented by the letters G. C. D.

ORAL EXERCISES

State the H. C. F. of the expressions in each of the following exercises.

1. 12 and 18.
2. 16 and 24.
3. x^2y and xy^2.
4. a^4b^4 and a^3b^3.
5. a^2b^3 and ab^2.
6. $x^2y^2z^3$ and xy^2z^2.
7. r^3sy^4, r^2sy, and rs^2y.
8. $2\,m^2n$, $3\,mn^2$, and $6\,mnp$.

NOTE. The H. C. F. of several monomials is most easily found by picking out the lowest power of each letter and multiplying them together.

Thus, in finding the H. C. F. of $x^2y^3z^2$, $x^4y^4z^3$ and x^3y^5z, the lowest power of x is x^2, the lowest power of y is y^3 and the lowest power of z is z. Therefore, the H. C. F. is $x^2 \cdot y^3 \cdot z$, or simply x^2y^3z. *Ans.*

By use of this Note state the H. C. F. in each of the following exercises.

9. x^2y^2z, $x^3y^3z^2$, and xy^2z^3.

10. m^2n^3q, mn^2q and mn.

11. $p^2q^3r^4s$ and p^3q^2r.

12. $2f^2g^2h$, g^3h^2i, and $3hi^2j$.

13. $4x^3y$ and $8y^2z$.

14. axy and bx^2y^2z.

State the H. C. F. in each of the following exercises (use Formulas of § 78).

15. a^2-b^2 and $a^2-2ab+b^2$.

16. a^2+ab and $a^2+2ab+b^2$.

17. a^2+2a+1 and a^2-1.

18. $a^2+7a+12$ and a^2-9.

* **19.** a^3-b^3 and $a-b$.

* **20.** a^3+b^3 and $a^2+2ab+b^2$.

84. General Rule for H. C. F. From what we have seen about the H. C. F. we may state the following general rule.

RULE FOR FINDING THE H. C. F. OF TWO OR MORE EXPRESSIONS.

Resolve each expression into its simplest factors.

Find the product of all the common factors, taking each factor the least number of times it occurs in any of the given expressions.

This product is the required H. C. F. of the given expressions.

WRITTEN EXERCISES

Find the H. C. F. of the expressions in each of the following exercises. Check your answer by showing that it is contained exactly in each of the given expressions. (See § 83.)

1. x^2-y^2 and $x^2-2xy+y^2$.

2. a^2+4a+3 and $a^2+8a+15$.

3. a^2+4a+4 and $a^2-6a-16$.

4. x^2-9 and x^2-x-6.

5. r^2-2r, r^2-r-2, and r^2-3r+2.

6. $y^4+3y^3+2y^2$, y^3+y^2, and $y^4+7y^3+6y^2$.

7. a^2-1, a^2-2a+1, and $a^2-13a+12$.

8. $1-5x$, $1-10x+25x^2$, and $1-25x^2$.

9. $3b^2-33b$ and $b^2-7b-44$.

10. $3y^3-y$, $3y^3-6y^2+3y$, and $6y^3+12y^2-15y$.

11. $a-b$, $(a-b)^2$, and $(a-b)^3$.

12. $a^2+2a-15$ and a^2-4a+3.

13. $2a^2+4a$ and $4a^3+12a^2+8a$.

14. $x^4-x^2y^2$ and x^2y+xy^2.

15. $a^4-a^3-2a^2$, $a^4-2a^3-3a^2$, and $a^4-3a^3-4a^2$.

16. a^4-2a^2+1 and a^2-2a+1.

17. a^2b-b^3, $ab+b^2$, and $a^2b^2-b^4$.

18. r^2-25, $r^2-7r+10$, and r^2-r-20.

19. $x^2-(y+1)^2$, $y^2-(x+1)^2$, and $1-(x+y)^2$.

20. x^2-y^2 and $x^3+x^2y+xy^2+y^3$.

[HINT. To factor the second expression, see § 54.]

21. $5b+20$ and $5ab+20a-2b-8$.

22. $y^2-11y+30$ and $yz-5z+y^2-5y$.

23. $3r^5+9r^4-3r^3$, $5r^2s^2+15rs^2-5s^2$, and $7ar^2+21ar-7a$.

24. $3b^3-3b$, $3b^3-6b^2+3b$, and $6b^3+12b^2-18b$.

25. $(1-x)^2$, x^2-1, and x^2-2x+1.

* **26.** a^3-b^3, a^2-b^2, and $a-b$.

* **27.** a^3+b^3 and $(a+b)^2$.

* **28.** x^3+1 and x^2-x+1.

* **29.** $(m+n)^2$, m^3+n^3, and m^2-n^2.

* **30.** x^3-1, $x^2-10x+9$, and x^2-x.

PART II. LOWEST COMMON MULTIPLE

85. Multiples. If one number is exactly divisible by another, the first is called a ***multiple*** of the second.

Thus, 12 is a multiple of 4. Likewise, in algebra, x^2y^2 is a multiple of x; it is also a multiple of y and of xy. Similarly, $a^2-4\,b^2$ is a multiple of $a-2\,b$ (see Formula VII) and of $a+2\,b$.

86. Common Multiples. In arithmetic a number which is exactly divisible by two or more other numbers is called a *common multiple* of them.

Thus, 48 is a common multiple of 6 and 12. Likewise, in algebra, an expression which is exactly divisible by two or more other expressions is called a *common multiple* of them.

Thus, $5\,x^2y^2$ is a common multiple of x and y; it is also a common multiple of 5 and x. Similarly, a^2-b^2 is a common multiple of $a-b$ and $a+b$.

Again, $3\,x^2yz^3$ is a common multiple of x, y, and z.

87. Lowest Common Multiple (L. C. M.). The lowest multiple common to two or more numbers is called in arithmetic their ***lowest (or least) common multiple***. It contains fewer prime factors than any other common multiple that the numbers can have.

Thus, of all the multiples common to 2 and 3 (such for example as 6, 12, 18, 24, 30, etc.) the lowest is 6; that is, 6 contains only the *two* prime factors 2 and 3, while all the other multiples in the list contain more. Likewise, the lowest common multiple of 12 and 18 is 36. (Why?)

Similarly, in algebra, we say that the ***lowest common multiple*** of two (or more) expressions is that multiple of them which contains the fewest possible prime factors. It is usually denoted by the abbreviation ***L. C. M.***

Thus, the L. C. M. of $4\,a$ and a^2b is $4\,a^2b$. Likewise, the L. C. M. of a and $a-4$ is $a(a-4)$, or $a^2-4\,a$. Again, the L. C. M. of the *three* expressions a^2b, axy and x^2yz^2 is $a^2bx^2yz^2$.

88. Important Property of the L. C. M. It is to be carefully observed that the L. C. M. of numbers or expressions must be exactly divisible by each of them, as follows from § 87. For example, 12, which is the L. C. M. of 2, 3, 4, and 6, contains 2 six times, 3 four times, etc. Likewise, the L. C. M. of ab and $5\,b^2c$ is $5\,ab^2c$, and this contains both ab and $5\,b^2c$ *exactly*. Explain this.

89. To Find the L. C. M. The way in which the L. C. M. of several numbers is found is illustrated below.

EXAMPLE 1. Find the L. C. M. of 24, 36, and 60.

SOLUTION. We write

$$24=2\cdot 2\cdot 2\cdot 3,$$
$$36=2\cdot 2\cdot 3\cdot 3,$$
$$60=2\cdot 2\cdot 3\cdot 5.$$

The L. C. M. is therefore $2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5$, or 360, since this contains all the factors of each of the three given numbers, and it is the *least* number that does so.

Similarly, the L. C. M. of several expressions in algebra is found in the manner illustrated below.

EXAMPLE 2. Find the L. C. M. of $10\,a^2b$, $16\,a^2b^3$, and $20\,a^3b^4$.

SOLUTION. We write

$$10\,a^2b=2\cdot 5\cdot a^2\cdot b,$$
$$16\,a^2b^3=2\cdot 2\cdot 2\cdot 2\cdot a^2b^3,$$
$$20\,a^3b^4=2\cdot 2\cdot 5\cdot a^3\cdot b^4.$$

The L. C. M. is thus seen to be $2\cdot 2\cdot 2\cdot 2\cdot 5\cdot a^3b^4$, or $80\,a^3b^4$, since this contains *all* the factors of each of the three given expressions, and at the same time it is made up of fewer factors than any other similar expression that can be found.

EXAMPLE 3. Find the L. C. M. of $a^4-10\,a^2+9$ and $a^2-4\,a+3$.

SOLUTION.

$$a^4-10\,a^2+9=(a^2-9)(a^2-1)=(a-3)(a+3)(a-1)(a+1),$$
$$a^2-4\,a+3=(a-3)(a-1).$$

The L. C. M. is, therefore, $(a-3)(a+3)(a-1)(a+1)$. *Ans.*

ORAL EXERCISES

State the L. C. M. of the expressions in each of the following exercises.

1. x^2y^3 and xy^3.
2. $4\,ab^2$ and $6\,ab$.
3. $6\,a^2c^2$ and $9\,ac^3$.
4. $8\,x^2$ and $4\,xy$.
5. $5\,ay^3$ and $10\,a^3y$.
6. xyz and xyw.
7. a^2bc^2 and ab^2c^3.
8. $12\,a^2$ and $6\,b^2$.
9. $a(b+c)$ and a^2.
10. $(a+b)^2$ and $a+b$.
11. x^3yz, xy^3z, and x^2y^2z.
12. ab, bc, and cd.
13. $3\,xy$, $6\,yz$, and $8\,xyz$.
14. $5\,ab^2$, $10\,bc^2$, and $4\,abc$.
15. $12\,ax^2y$, $4\,xy^2$, and $9a^2x^2y^2$.

90. General Rule for L. C. M. From what we have seen in § 89 concerning the L. C. M., we may now state the following general rule.

RULE FOR FINDING THE L. C. M. OF TWO OR MORE EXPRESSIONS.

Resolve each expression into its simplest factors.

Find the product of all the different factors, taking each factor the greatest number of times it occurs in any of the given expressions.

This product is the required L. C. M. of the given expressions.

WRITTEN EXERCISES

Find the L. C. M. of the expressions in each of the following exercises.

1. ab, a^2b, and ab^2.
2. x^2y^2, x^3y^3, and x^5y^2.
3. $4\,x^2$, $6\,xy$, and $12\,x^2y^2$.
4. $7\,x^2$, $14\,xy$, and $3\,y^2$.
5. $x+y$ and $ax+ay$.
6. x^2+xy and $x+y$.
7. $abc+abd$ and $c+d$.
8. x^2-y^2 and $x^2+2\,xy+y^2$.

9. x^2-y^2 and x^2+y^2.

10. x^2+2x and x^2-4.

11. $3y+2$ and $9y+6$.

12. $x-1$, $x+1$, and x^2-1.

13. a^2-1 and a^2-a-2.

14. a^2-3a+2 and $a-2$.

15. $(x-y)^3$ and $x^2-2xy+y^2$.

16. a^2+5a+6 and $a^2+7a+12$.

17. a^2+4a+3 and a^2+3a+2.

18. $3r^2+15r+18$ and r^2+6r+8.

19. a^4-1, a^2-1, and $a-1$.

20. $1-x^2$, $x-1$, and $(x-1)^2$.

21. $x^2+xy+xz+yz$ and $x^2+2xy+y^2$.

22. $(x+y)^3$ and x^2-y^2.

23. a^2+4a+4, a^2-4, $4-a^2$, and a^4-16.

24. x^2+x-42, $x^2-11x+30$, and $x^2+2x-35$.

25. $a^2-7a+10$, $a^2-10a+16$, and a^2-5a+6.

* 26. a^3-b^3 and a^2-b^2.

* 27. a^3+b^3 and a^2-b^2.

* 28. $8b^3-c^3$, $4b^2-4bc+c^2$, and $4b-2c$.

* 29. a^3b^3-27 and a^2b^2-ab-6.

* 30. $a^2x^3-a^2y^3$ and $ax^2-2axy+ay^2$.

CHAPTER IX

FRACTIONS

91. Algebraic Fractions. Algebraic fractions are like arithmetic fractions. Just as $\frac{3}{5}$ means $3 \div 5$, so $\frac{a}{b}$ (or a/b) means $a \div b$. In the same way, $\frac{a+5}{6}$ means $(a+5) \div 6$.

The dividend is called the ***numerator***, and the divisor is called the ***denominator***. The numerator and denominator taken together are called the ***terms*** of the fraction.

The fraction a/b is read *a divided by b*, or *a over b*.

The student will soon see that fractions in algebra are subject to the same rules that govern fractions in arithmetic. They are reduced to lower or higher terms, added, subtracted, multiplied, and divided just as arithmetical fractions.

92. Principle. In arithmetic we often change the form of a fraction without changing its value. Such changes all depend upon the following principle.

PRINCIPLE. *The numerator and denominator of a fraction may be multiplied or divided by the same number without changing the* ***value*** *of the fraction.*

Thus, $\frac{3}{4}=\frac{3 \cdot 2}{4 \cdot 2}=\frac{6}{8}$. Likewise, $\frac{8}{12}=\frac{8 \div 4}{12 \div 4}=\frac{2}{3}$.

The same principle applies in algebra. Thus,

$$\frac{4a}{5}=\frac{4a \cdot a}{5 \cdot a}=\frac{4a^2}{5a^2}; \qquad \frac{6a}{10}=\frac{(6a)\div 2}{10 \div 2}=\frac{3a}{5};$$

$$\frac{a}{b}=\frac{a \cdot a}{b \cdot a}=\frac{a^2}{ab}; \qquad \frac{a+b}{(a+b)^2}=\frac{\cancel{(a+b)}}{(a+b)\cancel{(a+b)}}=\frac{1}{a+b}.$$

93. Signs in Fractions. There are three signs to be considered in a fraction: first, *the sign of the numerator;* second, *the sign of the denominator;* and third, *the sign of the fraction itself,* which is placed just before the dividing line.

Thus, in the fraction

$$+\frac{-3}{4},$$

the sign of the numerator is $-$, the sign of the denominator is $+$, while the sign of the fraction itself is $+$. Again, in the fraction

$$-\frac{+a}{-b},$$

the sign of the numerator is $+$, that of the denominator is $-$, and that of the fraction itself is $-$.

Since a fraction is merely an indicated division, the law of signs for division (§ 70) must hold for all fractions.

Thus we have,

$$+\frac{+12}{+6}=+2; \qquad +\frac{-12}{-6}=+2;$$

$$-\frac{-12}{+6}=-(-2)=+2; \qquad -\frac{+12}{-6}=-(-2)=+2.$$

Hence we may write

$$+\frac{+12}{+6}=+\frac{-12}{-6}=-\frac{-12}{+6}=-\frac{+12}{-6},$$

since each of these forms is equal to 2.

Likewise, in all cases we have

$$+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}.$$

From these examples it appears that *if any two of the three signs of a fraction are changed, the value of the fraction is* ***not*** *changed.*

Care must be taken, however, in changing the sign of the numerator or denominator of a fraction if *polynomials* are

present. Thus, if the numerator is a polynomial, we can change the sign of the whole numerator only by changing the sign of *every* term in it. A similar statement applies when the denominator is a polynomial.

For example, we have

$$+\frac{a+2b+c}{2a-3b-2c}=-\frac{-a-2b-c}{2a-3b-2c}=-\frac{a+2b+c}{-2a+3b+2c}.$$

Observe carefully the reason for every change of sign here.

ORAL EXERCISES

1. State three other ways of writing the fraction $-\frac{+3}{-5}$.

2. State three other ways of writing $\frac{3}{-5}$.

[HINT. Remember that $\frac{3}{-5}$ means $+\frac{+3}{-5}$ (§ 19).]

3. State three other ways of writing each of the following fractions.

$$-\frac{2}{3},\quad \frac{9}{10},\quad \frac{-4}{5},\quad \frac{2a}{3},\quad -\frac{a+b}{c-d},\quad \frac{2x+y}{z-1},\quad \frac{x^2+x-1}{3xy+7}.$$

Change the following into fractions having no negative signs in either their numerator or denominator.

4. $\frac{-5}{8}$.

5. $\frac{6}{-11}$.

6. $\frac{-8}{-9}$.

7. $\frac{-b}{c}$.

8. $\frac{a}{-b}$.

9. $\frac{-c}{-d}$.

10. $\frac{a+b}{-c}$.

11. $\frac{-x}{x+y}$.

12. $\frac{2b}{-c-3y}$.

13. $-\frac{-x-5}{y+3}$.

14. $\frac{x+2y+z}{-3x^2-y-z^2}$.

15. $\frac{-mn-m^3-n^2}{6pq+r+s^2}$.

94. Reduction of Fractions to Lowest Terms. A fraction is reduced to its lowest terms when its numerator and denominator have no common factor except 1.

Thus, each of the fractions

$$\frac{5}{6},\ \frac{x}{y},\ \frac{2\,a}{3\,b},\ \text{and}\ \frac{a+b}{a-b}$$

is in its lowest terms, but

$$\frac{4}{6},\ \frac{xy}{y^2},\ \text{and}\ \frac{a^2-b^2}{a+b},$$

are *not* in their lowest terms. Explain why not in each case.

To reduce a fraction to its lowest terms, *factor the numerator and denominator and then divide both numerator and denominator by all their common factors.*

EXAMPLE 1. Reduce $\frac{25\,a^2b^3x}{35\,a^3bx}$ to its lowest terms.

SOLUTION.

$$\frac{25\,a^2b^3x}{35\,a^3bx}=\frac{\cancel{5}\cdot 5\cdot \cancel{a}\cdot \cancel{a}\cdot \cancel{b}\cdot b\cdot b\cdot \cancel{x}}{\cancel{5}\cdot 7\cdot \cancel{a}\cdot \cancel{a}\cdot a\cdot \cancel{b}\cdot \cancel{x}}=\frac{5\,b^2}{7\,a}.\quad \textit{Ans.}$$

Thus, division in algebra may be carried out by canceling, just as in arithmetic. Doing this is equivalent to dividing both numerator and denominator by *all* their common factors, which is the same as dividing them by their H. C. F.

EXAMPLE 2. Reduce $\frac{a^2-11\,a+24}{a^2-5\,a+6}$ to its lowest terms.

SOLUTION.

$$\frac{a^2-11\,a+24}{a^2-5\,a+6}=\frac{(a-8)\cancel{(a-3)}}{(a-2)\cancel{(a-3)}}=\frac{a-8}{a-2}.\quad \textit{Ans.}$$

Observe that only factors that are common to both numerator and denominator can be canceled. It is a common error on the part of students to cancel *terms* instead of factors. Thus, in the fraction

$$\frac{a+x}{a+b},$$

the a's cannot be canceled, for, though a is here a common term, it is *not* a common factor of numerator and denominator.

Another common error is to write 0 for the result whenever *all* the factors of numerator and denominator cancel each other. The answer in such cases is always 1 instead of 0. Thus

$$\frac{(a+b)^2}{a^2+2\,ab+b^2}=\frac{\overset{1}{\cancel{(a+b)}}\,\overset{1}{\cancel{(a+b)}}}{\cancel{(a+b)}\,\cancel{(a+b)}}=1\times1=1. \quad Ans.$$

The principle to be remembered here is that any number when divided by *itself* gives 1. Thus,

$$\frac{2}{2}=1,\ \frac{4}{4}=1,\ \frac{-3}{-3}=1,\ \frac{a}{a}=1,\ \frac{a+b}{a+b}=1,\ \text{etc.}$$

ORAL EXERCISES

Reduce each of the following fractions to its lowest terms.

1. $\dfrac{8}{12}$.
2. $\dfrac{9}{12}$.
3. $\dfrac{10}{14}$.
4. $\dfrac{25}{30}$.
5. $\dfrac{4\,a}{12}$.
6. $\dfrac{8\,r}{16}$.
7. $\dfrac{20\,b}{50}$.
8. $\dfrac{18\,a^2}{30}$.
9. $\dfrac{ab}{ax}$.
10. $\dfrac{xy}{x^2y^2}$.
11. $\dfrac{x^2y^2}{x^3y^3}$.
12. $\dfrac{xy^2}{x^2y}$.
13. $\dfrac{5\,xy}{10\,x^2y^2}$.
14. $\dfrac{36\,xr^3}{72\,x^3r^3}$.
15. $\dfrac{2\,abc}{abc}$.
16. $\dfrac{a^2b^2c^2}{a^2b^3c^2}$.

WRITTEN EXERCISES

Reduce each of the following fractions to its lowest terms. Starred (*) exercises require § 75.

1. $\dfrac{20\,x^2yz^2}{130\,ax^3yz^5}$.
2. $\dfrac{240\,xy^2z}{800\,by^2z}$.
3. $\dfrac{75\,abcd}{150\,a^2b^3c^4d^5}$.
4. $\dfrac{88\,a^6x^4b^2y}{121\,a^3b^7c^4}$.
5. $\dfrac{-25\,xyz^3}{-150\,x^3y^3}$.
6. $\dfrac{(a-b)^2}{a^2-2\,ab+b^2}$.
7. $\dfrac{8+8\,x+2\,x^2}{(x+2)^2}$.
8. $\dfrac{a^2-b^2}{a^2-2\,ab+b^2}$.
9. $\dfrac{a^2-b^2}{a^2+2\,ab+b^2}$.

10. $\dfrac{9x^2-y^2}{9x^2+6xy+y^2}$. 11. $\dfrac{a^2b^2+ab}{(ab+1)^2}$. 12. $\dfrac{2x^2y-2x^2}{3x^3y-3x^3}$.

13. $\dfrac{4x^2-y^2}{y-2x}$.

[HINT. Change the sign of the denominator and the sign of the fraction. Then the fraction takes the form $-\dfrac{4x^2-y^2}{2x-y}$. See § 93.]

14. $\dfrac{s^2-6s+8}{s^2-5s+6}$.

15. $\dfrac{a^2b^2+10ab+21}{a^2b^2+11ab+24}$.

16. $\dfrac{a^2-15ab-34b^2}{a^2-6ab-16b^2}$.

17. $\dfrac{m^4-m^2n^2}{m^2-n^2}$.

18. $\dfrac{xy+y^2}{xz+yz}$.

19. $\dfrac{a^2-a-2}{-a^2+4}$.

[HINT. See Ex. 13.]

20. $\dfrac{x^2+x-6}{15+2x-x^2}$.

21. $\dfrac{r^2-2rs-15s^2}{r^2+8rs+15s^2}$.

22. $\dfrac{r^4-6r^2+5}{r^2-6r+5}$.

23. $\dfrac{b^3+7b^2-8b}{b^4+7b^3-8b^2}$.

* 24. $\dfrac{x^2-y^2}{x^3-y^3}$.

* 25. $\dfrac{-x^3+27}{x^2+6x-27}$.

[HINT. See Ex. 13.]

* 26. $\dfrac{-27-a^3}{a^2-9}$.

* 27. $\dfrac{r^2-2r^4+r^6}{-r^6+r^2}$.

28. $\dfrac{m^4-n^4}{m+n}$.

29. $\dfrac{x^2-4y^2}{(x-2y)^2}$.

For further exercises on this topic, see the review exercises, p. 161, *and Appendix, p.* 301.

95. Reduction of Fractions to Common Denominator. Such fractions as $\frac{2}{3}$, $\frac{5}{3}$, $\frac{17}{3}$ are said to have a ***common denominator***; that is, the denominator of each is the same, being 3. Similarly, the fractions

$$\frac{a}{b},\ \frac{3}{b},\ \frac{4y}{b},\ \frac{x^2z}{b}$$

have the common denominator b; and the fractions

$$\frac{a}{a+b},\ \frac{6g}{a+b},\ \frac{x^2+y^2}{a+b}$$

have the common denominator $a+b$.

When two or more fractions do *not* have a common denominator, it is always possible to so change their forms as to make them have one.

Thus, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{18}$ can be changed (see § 92) into the forms $\frac{24}{36}$, $\frac{27}{36}$, and $\frac{10}{36}$. As they now appear, they have the common denominator 36.

Similarly, the three fractions $\frac{a}{b}$, $\frac{a}{3b^2}$, and $\frac{x}{by}$ may be put into the forms $\frac{3aby}{3b^2y}$, $\frac{ay}{3b^2y}$, and $\frac{3bx}{3b^2y}$. Explain. As they now appear, they have the common denominator $3b^2y$.

96. Lowest Common Denominator (L. C. D.). The two fractions $\frac{1}{2}$ and $\frac{2}{3}$ may be made to have a common denominator in many different ways. Thus we may write them as $\frac{3}{6}$ and $\frac{4}{6}$, or as $\frac{6}{12}$ and $\frac{8}{12}$, or as $\frac{9}{18}$ and $\frac{12}{18}$, etc. Of all these ways the most important one (as we shall see) is that in which the common denominator is *lowest*. In other words, the most important way of writing $\frac{1}{2}$ and $\frac{2}{3}$ so that they shall have a common denominator is $\frac{3}{6}$ and $\frac{4}{6}$, respectively. Note that as thus written, the common denominator, 6, is simply the lowest (or least) common multiple (§ 87) of the two given denominators 2, 3. This one denominator, 6, is called the ***lowest common denominator*** of $\frac{1}{2}$ and $\frac{2}{3}$.

In general, two or more fractions may always be made to appear with their lowest common denominator in the way just mentioned. We have only to find the L. C. M. of the various denominators, and then change each fraction (by § 92) so that it will have this L. C. M. as its new denominator.

For example, in dealing with the fractions

$$\frac{2}{3}, \frac{3}{4}, \frac{5}{9}, \frac{7}{12},$$

the denominators are 3, 4, 9, and 12. The L. C. M. of these is 36. So the given fractions, when written with lowest common denominator, are

$$\frac{24}{36}, \frac{27}{36}, \frac{20}{36}, \frac{21}{36}.$$

What has just been said applies word for word to algebra, as is illustrated in the following examples.

EXAMPLE 1. Change the fractions

$$\frac{x}{4\,yz} \text{ and } \frac{2\,r}{y^2z^3}$$

into equal fractions having their lowest common denominator.

SOLUTION. The two denominators are $4\,yz$ and y^2z^3 and the L. C. M. of these (as found by § 89) is $4\,y^2z^3$.

Writing the first fraction with the denominator $4\,y^2z^3$ (by multiplying both its numerator and denominator by yz^2) gives

$$\frac{xyz^2}{4\,y^2z^3}.$$

Similarly, writing the second fraction with the same denominator, $4\,y^2z^3$ (by multiplying both its numerator and denominator by 4) gives

$$\frac{8\,r}{4\,y^2z^3}.$$

Hence these are the required new forms.

EXAMPLE 2. Change the fractions

$$\frac{a+2}{a^2-9} \text{ and } \frac{a+5}{a^2+9\,a+18}$$

into equal fractions having their lowest common denominator.

SOLUTION. First factor the given denominators, thus writing the two fractions in the forms

$$\frac{a+2}{(a+3)(a-3)},\ \frac{a+5}{(a+6)(a+3)}.$$

The L. C. M. of these denominators is therefore (by § 89) seen to be

$$(a+3)(a-3)(a+6).$$

To give the first fraction this denominator, we must multiply its numerator and denominator by $(a+6)$.

Similarly, to give the second fraction the new denominator we must multiply both its numerator and denominator by $(a-3)$.

The desired new forms are therefore

$$\frac{(a+2)(a+6)}{(a+3)(a-3)(a+6)},\ \frac{(a+5)(a-3)}{(a+3)(a-3)(a+6)}.\quad \textit{Ans.}$$

EXAMPLE 3. Change the fractions

$$\frac{x-1}{x^2-4},\quad \frac{2y}{x^2+x-6},\quad \text{and}\quad \frac{z}{x^2-x-6}$$

into equal fractions having their lowest common denominator.

SOLUTION. Factoring the denominators, the fractions become

$$\frac{x-1}{(x-2)(x+2)},\quad \frac{2y}{(x-2)(x+3)},\quad \frac{z}{(x+2)(x-3)}.$$

The L. C. M. of the denominators is therefore $(x-2)(x+2)(x+3)(x-3)$. Giving each of the fractions this denominator, they become

$$\frac{(x-1)(x+3)(x-3)}{(x-2)(x+2)(x+3)(x-3)},\quad \frac{2y(x+2)(x-3)}{(x-2)(x+2)(x+3)(x-3)},$$

$$\frac{z(x-2)(x+3)}{(x-2)(x+2)(x+3)(x-3)}.\quad \textit{Ans.}$$

97. From these examples we have the following rule.

RULE FOR REDUCING FRACTIONS TO THEIR LOWEST COMMON DENOMINATOR. *Find the L. C. M. of the denominators. Then multiply both the numerator and denominator of each fraction by such an expression as will make that fraction have this L. C. M. as its denominator.*

ORAL EXERCISES

First state the new *denominator* for each of the following fractions in order that they may have their lowest common denominator; then state the new forms for the fractions themselves. (Use the rule in § 97.)

1. $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$.

2. $\dfrac{a}{6}$ and $\dfrac{b}{a}$.

3. $\dfrac{3}{a}$ and $\dfrac{2}{b}$.

4. $\dfrac{1}{ab}$ and $\dfrac{4}{a^2}$.

5. $\frac{a}{x}$, $\frac{2b}{3x}$, and $\frac{4}{6x}$.

6. $\frac{rs}{abx}$, $\frac{st}{a^2x^2}$, and $\frac{s}{a^2b^2x}$.

7. $\frac{1}{a+b}$ and $\frac{1}{a-b}$.

8. $\frac{3b}{x^2-y^2}$ and $\frac{2}{x-y}$.

9. $\frac{3k}{x+5}$ and $\frac{k}{x^2-2x-15}$.

10. $\frac{a}{(a+b)^2}$, $\frac{r}{1}$, and $\frac{1}{a+b}$.

11. $\frac{a}{b}$, $\frac{x}{a-b}$, and $\frac{y}{b-a}$.

[HINT. Write $\frac{y}{b-a}$ in the form $\frac{-y}{a-b}$.]

12. $\frac{s}{s-1}$ and $\frac{1}{s(1-s)}$.

WRITTEN EXERCISES

Write the fractions in each of the following exercises with lowest common denominator.

1. $\frac{x-y}{4xy^3}$ and $\frac{x+2y}{3x^2y}$.

2. $\frac{3+n^2}{2n}$ and $\frac{5+n^2}{3m}$.

3. $\frac{5}{r+6}$ and $\frac{5}{r-6}$.

4. $\frac{s}{s^2-2s+1}$ and $\frac{1}{1-s}$.

5. $\frac{4}{x+3}$ and $\frac{6}{x^2-12x-45}$.

6. $\frac{1}{s^2+7s+10}$, $\frac{1}{s^2+s-2}$, and $\frac{1}{s^2+4s-5}$.

7. $\frac{a}{r-s}$, $\frac{b}{s-r}$, and $\frac{c}{-s-r}$.

8. $\frac{a}{(a+b)^2}$, $\frac{r}{1}$, and $\frac{1}{a^2-b^2}$.

9. $\frac{1}{b^2+5b+6}$ and $\frac{1}{2(b^2+6b+9)}$.

10. $\frac{m-2}{m^2-2m-8}$, $\frac{m-1}{m^2-3m-10}$, and $\frac{m+3}{m^2-9m+20}$.

98. Addition and Subtraction of Fractions. In adding and subtracting fractions in algebra we proceed as in arithmetic.

Thus, to add $\frac{2}{3}$ and $\frac{3}{4}$, we reduce them to $\frac{8}{12}$ and $\frac{9}{12}$, and the sum is then $\frac{17}{12}$, or $1\frac{5}{12}$. Instead of reducing the fractions to the common denominator 12, we might reduce them to the common denominator 24; that is, to $\frac{16}{24}$ and $\frac{18}{24}$. In this case the sum is $\frac{34}{24}$, or $1\frac{5}{12}$ as before. But it is better to do the first way; that is, to reduce to the *lowest* common denominator, and then add.

In the same way, in adding $\frac{a}{c}$ and $\frac{b}{d}$, we first reduce them to *lowest common denominator*, thus getting $\frac{ad}{cd}$ and $\frac{bc}{cd}$. Then the sum is

$$\frac{ad+bc}{cd}.$$

EXAMPLE 1. Simplify $\frac{2x}{3}+\frac{3x}{5}+\frac{7x}{12}$.

SOLUTION. Reducing the given fractions to their lowest common denominator, we have

$$\frac{2x}{3}+\frac{3x}{5}+\frac{7y}{12}=\frac{40x}{60}+\frac{36x}{60}+\frac{35y}{60}=\frac{40x+36x+35y}{60}$$
$$=\frac{76x+35y}{60}. \quad \textit{Ans.}$$

EXAMPLE 2. Subtract $\frac{2a+3b}{3}$ from $\frac{6a-5b}{6}$.

SOLUTION. Reducing the given fractions to their lowest common denominator, we have

$$\frac{6a-5b}{6}-\frac{2a+3b}{3}=\frac{6a-5b}{6}-\frac{4a+6b}{6},$$
$$=\frac{6a-5b-(4a+6b)}{6},$$

where the parentheses appear because the *whole* expression $4a+6b$ is to be subtracted. Removing the parentheses, we find

$$\frac{6a-5b-4a-6b}{6}$$
$$=\frac{2a-11b}{6}. \quad \textit{Ans.}$$

EXAMPLE 3. Simplify $\dfrac{4}{a-1}-\dfrac{a-2}{a+1}+\dfrac{3a^2}{a^2-1}$.

SOLUTION. $\dfrac{4}{a-1}-\dfrac{a-2}{a+1}+\dfrac{3a^2}{a^2-1}$

$$=\frac{4a+4}{a^2-1}-\frac{a^2-3a+2}{a^2-1}+\frac{3a^2}{a^2-1} \qquad \text{(Explain)}$$
$$=\frac{4a+4-(a^2-3a+2)+3a^2}{a^2-1} \qquad \text{(Explain)}$$
$$=\frac{2a^2+7a+2}{a^2-1}. \quad \textit{Ans.}$$

99. Rule for Addition and Subtraction of Fractions. From the examples of § 98, we may state the following rule.

RULE FOR ADDING OR SUBTRACTING FRACTIONS. *Reduce the fractions to equal fractions having their lowest common denominator. Add or subtract each numerator according to the sign before the fraction and write the result over the lowest common denominator.*

Reduce the resulting fraction to its lowest terms if necessary.

ORAL EXERCISES

Simplify each of the following expressions.

1. $\frac{2}{3}+\frac{3}{4}$.
2. $\frac{5}{6}-\frac{4}{9}$.
3. $\frac{2}{3}+\frac{5}{6}-\frac{3}{4}$.
4. $\frac{5}{12}-\frac{1}{4}+\frac{2}{3}$.
5. $\frac{4}{25}-\frac{1}{5}-\frac{3}{10}$.
6. $\frac{1}{5}+\frac{1}{6}-\frac{1}{3}$.
7. $\dfrac{2a}{3}+\dfrac{3a}{4}-\dfrac{a}{2}$.
8. $\dfrac{3x}{10}-\dfrac{4x}{15}$.
9. $\dfrac{2t}{5}+\dfrac{t-3}{6}$.
10. $\dfrac{a}{b}+\dfrac{c}{d}$.
11. $\dfrac{2r}{x}-\dfrac{3s}{y}$.
12. $\dfrac{a+b}{3}-\dfrac{a-b}{4}$

WRITTEN EXERCISES

Simplify each of the following expressions.

1. $\frac{3x}{4}-\frac{x}{5}$.

2. $\frac{y-1}{4}+\frac{2y+5}{2}$.

3. $\frac{4x+7}{15}-\frac{2(x-1)}{20}$.

4. $\frac{x-4}{3}-\frac{2(1-x)}{6}+\frac{x}{8}$.

5. $a-\frac{a}{2}+\frac{2a}{3}$.

6. $r-\frac{3(4-r)}{4}$.

7. $1+\frac{s-1}{2}$.

8. $x-3-\frac{2(x+4)}{3}$.

9. $\frac{x-y}{2}-\frac{y-x}{2}$.

10. $\frac{1}{2}(a+b)+\frac{2}{3}(a-b)$.

11. $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$.

12. $\frac{3}{ab^2}-\frac{2}{ab}+\frac{4}{a^2}$.

3. $\frac{1}{2a-2}-\frac{1}{3(a+1)}+\frac{a-4}{3a-3}$.

SOLUTION. The least common denominator is $6(a^2-1)$. Reducing the fractions to this denominator, we have

$$\frac{3a+3}{6(a^2-1)}-\frac{2a-2}{6(a^2-1)}+\frac{2a^2-6a-8}{6(a^2-1)}=\frac{2a^2-5a-3}{6(a^2-1)}. \quad \textit{Ans.}$$

14. $\frac{1}{x-y}+\frac{1}{x+y}$.

15. $\frac{1}{x+y}+\frac{1}{y-x}$.

16. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

17. $\frac{x-1}{(x+1)^2}-\frac{x}{x+1}$.

18. $\frac{r+1}{r^2-2r-8}-\frac{r}{r-4}+\frac{1}{r+2}$

19. $\frac{1}{x^2-4x-5} \quad \frac{1}{x^2-6x+5}$

20. $\frac{a+b}{a^2+ab+b^2} \quad \frac{a-b}{a^2-ab+b^2}$

21. $\frac{a^2+b^2}{a^4+a^2b^2+b^4}+\frac{a^2-b^2}{a^4-a^2b^2+b^4}$.

For further exercises on this topic, see the review exercises, p. 161, *and Appendix, p.* 302.

100. Multiplication of Fractions. Fractions are *multiplied* in algebra as in arithmetic, *by taking the products of the numerators for a new numerator, and the products of the denominators for a new denominator.* Thus,

$$\frac{2}{3}\cdot\frac{5}{7}=\frac{2\cdot 5}{3\cdot 7}=\frac{10}{21}. \quad \textit{Ans.} \qquad \frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}=\frac{ac}{bd}. \quad \textit{Ans.}$$

EXAMPLE 1. Multiply $\dfrac{3\,a^2b}{4\,xy^4}$ by $\dfrac{2\,x^2y^5}{9\,a^2b^3}$.

SOLUTION. $\dfrac{3\,a^2b}{4\,xy^4}\cdot\dfrac{2\,x^2y^5}{9\,a^2b^3}=\dfrac{6\,a^2bx^2y^5}{36\,a^2b^3xy^4}=\dfrac{xy}{6\,b^2}.$ *Ans.*

As in arithmetic, factors common to the numerator and denominator should be canceled. To do this, the numerator and denominator should first be factored.

EXAMPLE 2. Multiply $\dfrac{a^2+2\,a-8}{a^2+6\,a}$ by $\dfrac{a^2+5\,a-6}{a^2+a-12}$.

SOLUTION.

$$\frac{a^2+2\,a-8}{a^2+6\,a}\cdot\frac{a^2+5\,a-6}{a^2+a-12}=\frac{\cancel{(a+4)}(a-2)}{a\cancel{(a+6)}}\cdot\frac{\cancel{(a+6)}(a-1)}{\cancel{(a+4)}(a-3)}=\frac{(a-2)(a-1)}{a\,(a-3)}.$$

Ans.

101. NOTE. An integer may be regarded as a fraction with the denominator 1.

Thus, $16\cdot\dfrac{r}{9}=\dfrac{16}{1}\cdot\dfrac{r}{9}=\dfrac{16\cdot r}{1\cdot 9}=\dfrac{16\,r}{9}$; $a\cdot\dfrac{b}{c}=\dfrac{a}{1}\cdot\dfrac{b}{c}=\dfrac{a\cdot b}{1\cdot c}=\dfrac{ab}{c}$.

ORAL EXERCISES

State the result in each of the following multiplications.

1. $3\cdot\frac{2}{5}$.

[HINT. See Note, § 101.]

2. $a\cdot\dfrac{r}{s}$.

3. $\frac{2}{3}\cdot\frac{3}{4}$.

4. $\dfrac{a}{b}\cdot\dfrac{b}{c}$.

5. $\dfrac{2\,ab}{3\,xy}\cdot\dfrac{4\,x^2}{5\,a^2}$.

6. $\dfrac{4\,mn}{3\,rs}\cdot\left(-\dfrac{12\,rs}{m^2n^2}\right)$.

7. $-\dfrac{6\,a}{18\,by}\cdot\dfrac{10\,b^2}{8\,a^2}$.

8. $\left(-\dfrac{4\,m}{9\,x^2y^2}\right)\cdot\left(-\dfrac{6\,x^3y^3}{8\,m^2}\right)$.

WRITTEN EXERCISES

Carry out each of the following indicated multiplications.

1. $\frac{3}{4} \cdot \frac{7}{12} \cdot \frac{8}{15}$.

2. $\frac{3^2}{5^2} \cdot \frac{5^4}{3^4}$.

3. $\frac{x^2y^2}{36} \cdot \frac{60}{x^3y^3}$.

4. $\frac{m^2n}{rs^2} \cdot \frac{mn^2}{r^3s} \cdot \frac{r^4s^4}{m^3n^3}$.

5. $\frac{6\,x}{7\,y} \cdot \frac{8\,y^2}{9\,x^3} \cdot \frac{12\,y}{15\,x}$.

6. $\frac{-2\,a}{c^2} \cdot \frac{-3\,c}{4\,a} \cdot \frac{4\,a}{5\,c}$.

7. $\frac{a^2-b^2}{a^2+b^2} \cdot \frac{2}{a+b}$.

[HINT. See Ex. 2, § 100.]

8. $\frac{x^2-y^2}{x^2-2\,xy+y^2} \cdot \frac{x-y}{x+y}$.

Check your result for $x=2$, $y=1$.

9. $\frac{x+5}{x-4} \cdot \frac{16-x^2}{25-x^2}$.

10. $\frac{2\,a+2\,b}{3\,a-3\,b} \cdot \frac{4\,(a-b)}{5\,(a+b)}$.

11. $\frac{5\,x-5\,y}{2\,x^2+4\,xy+2\,y^2} \cdot \frac{x+y}{5}$.

12. $\frac{a+2}{a+3} \cdot \frac{a-5}{a+6}$.

13. $\frac{x^2+2\,x-3}{x^2-2\,x-3} \cdot \frac{x^2-2\,x-3}{x^2+4\,x+3}$.

14. $\frac{r+2}{2\,s-3} \cdot \frac{s-1}{2\,r+2} \cdot \frac{4\,s-6}{r+2}$.

15. $\frac{a^2+3\,a-10}{a^2+2\,a-3} \cdot \frac{a^2+7\,a+12}{a^2-9\,a+14}$.

16. $x \cdot \frac{x^3}{y^3} \cdot \frac{x-y}{x^4}$.

17. $(x+y) \cdot \frac{x-y}{x+y} \cdot \frac{x}{x^2-y^2}$.

18. $\frac{(a+2)(a+3)}{(a+4)} \cdot \frac{(a+3)(a+4)}{(a+2)} \cdot \frac{1}{(a+3)^2}$.

19. $\frac{a+b}{a-b} \cdot (a^2-2\,ab+b^2) \cdot \frac{a^2+b^2}{a^2-b^2}$.

20. $\left(a+\frac{b}{c}\right)\left(a-\frac{b}{c}\right)$.

21. $\frac{\pi a^2-\pi}{a+1} \cdot \frac{1}{\pi a-\pi}$.

22. $\frac{x^2-5\,x+6}{x^2-7\,x+12} \cdot \frac{x^2-6\,x+8}{x^2-8\,x+15} \cdot \frac{x^2-9\,x+20}{x^2-7\,x+10}$.

23. $\frac{y^2-1}{1+y} \cdot \frac{y^2+5\,y+4}{y^2+2\,y+1} \cdot \frac{1}{1-y}$.

24. $\frac{2+s-s^2}{s^2-s-2} \cdot \frac{s^2-1}{4\,s^2-1}$.

For further exercises on this topic, see the review exercises, *p.* 161, *and Appendix, p.* 302.

102. Division of Fractions. In algebra we divide one fraction by another as we do in arithmetic, *by inverting the divisor and proceeding as in multiplication.*

Thus, $$\frac{15}{16} \div \frac{3}{4} = \frac{15}{16} \cdot \frac{4}{3} = \frac{\overset{5}{\cancel{15}} \cdot \cancel{4}}{\underset{4}{\cancel{16}} \cdot \cancel{3}} = \frac{5}{4}. \quad \textit{Ans.}$$

Again, $$\frac{ab}{4\,xy} \div \frac{ab}{4\,y} = \frac{ab}{4\,xy} \cdot \frac{4\,y}{ab} = \frac{\cancel{a} \cdot \cancel{b} \cdot \cancel{4} \cdot \cancel{y}}{\cancel{4} \cdot x \cdot \cancel{y} \cdot \cancel{a} \cdot \cancel{b}} = \frac{1}{x}. \quad \textit{Ans.}$$

In division, as in multiplication, the different numerators and denominators should be factored before canceling.

EXAMPLE 1. Divide $\dfrac{a^2-b^2}{a^2+2\,ab+b^2}$ by $\dfrac{a-b}{a^2+ab}$.

SOLUTION. $$\frac{a^2-b^2}{a^2+2\,ab+b^2} \div \frac{a-b}{a^2+ab} = \frac{\cancel{(a+b)}\cancel{(a-b)}}{\cancel{(a+b)}\cancel{(a+b)}} \cdot \frac{a\cancel{(a+b)}}{\cancel{(a-b)}} = a. \quad \textit{Ans.}$$

CHECK. If $a=1$, and $b=2$ we have

$$\frac{a^2-b^2}{a^2+2\,ab+b^2} \div \frac{a-b}{a^2+ab}$$

$$= \frac{1-4}{1+2\cdot 1\cdot 2+4} \div \frac{1-2}{1+1\cdot 2} = \frac{-3}{9} \div \frac{-1}{3} = \frac{-3}{9} \cdot \frac{3}{-1} = 1,$$

which is the same as the *answer* above (which was a), when $a=1$.

ORAL EXERCISES

Simplify each of the following expressions.

1. $\dfrac{4}{9} \div \dfrac{2}{3}$.

2. $\dfrac{5\,x}{16\,y} \div \dfrac{3\,x}{4\,y}$.

3. $\dfrac{10\,x^2}{11\,y} \div \dfrac{12\,x}{y^2}$.

4. $\dfrac{6\,ab^3}{5\,rs^2} \div \dfrac{2\,ab}{rs^2}$.

5. $\dfrac{7\,x^2y^3}{10\,axy^2} \div \dfrac{3\,xy}{4\,ay}$.

6. $\dfrac{4\,x}{5\,x^2y^2} \div \dfrac{4}{x^2}$.

7. $\dfrac{a^2b}{c} \div \dfrac{a^2b}{c}$.

8. $\dfrac{10\,x}{13\,y^2} \div \dfrac{20\,x}{3\,y}$.

WRITTEN EXERCISES

Simplify each of the following expressions.

1. $\frac{5\,a^2}{2\,by} \div 5.$

[HINT. We here have

$$\frac{5\,a^2}{2\,by} \div \frac{5}{1}.$$

See Note in § 101.]

2. $\frac{4\,a}{5\,x^2y^2} \div a.$

3. $\frac{3\,ab^2}{4\,r^2s} \div 3\,ab^2.$

4. $\frac{a^2+ab}{a^2-b^2} \div \frac{a^2+2\,ab+b^2}{a^2-2\,ab+b^2}.$

[HINT. Remember that each polynomial should first be factored.]

5. $\frac{a^2+2\,a}{a^2-2\,a} \div \frac{(a+2)^2}{(a-2)^2}.$

6. $\frac{a^2+6\,a+9}{a^2-9} \div \frac{a+3}{a-3}.$

7. $\frac{x^2+7\,x+12}{x^2+x-12} \div \frac{x+4}{x-3}.$

8. $\frac{a^2+9\,ab+20\,b^2}{a^2+2\,ab} \div \frac{a^2+4\,ab}{a^2+7\,ab+10\,b^2}.$

9. $\frac{3\,t^2+3\,s^2}{t-s} \div \frac{t^2+s^2}{t-s}.$

10. $\frac{r^4-s^4}{r^3-rs^2} \div \frac{r^2-s^2}{r^2+s^2}$

11. $\frac{x^2-1}{x^2-3\,x+2} \div \frac{x-1}{x-2}.$

12. $\frac{x^2-xy}{r-s} \div \frac{x^4-y^4}{(x-y)^2}.$

13. $\frac{2\,ab-b^2}{a(a+b)} \div \frac{a^2-b^2}{2\,a-b}.$

14. $\frac{x^2-121}{x^2-4} \div \frac{x+11}{x+2}.$

15. $\frac{4\,a^2-9\,b^2}{a^2-4} \div \frac{2\,a-3\,b}{a-2}.$

16. $\dfrac{a^2-1}{a^2-3a-10} \div \dfrac{a^2-12a+35}{a^2+3a+2}$.

17. $\dfrac{2xy-y^2}{xy+y^2} \div \dfrac{x^2-y^2}{x+y}$.

18. $\dfrac{r^2-y^2}{r} \div (r-y)$.

[HINT. See Ex. 1.]

19. $\left(a-\dfrac{2}{b}\right) \div \left(a+\dfrac{2}{b}\right)$.

$\left[\text{HINT.}\quad \left(a-\dfrac{2}{b}\right) \div \left(a+\dfrac{2}{b}\right) = \dfrac{ab-2}{b} \div \dfrac{ab+2}{b}.\right]$

20. $\left(2+\dfrac{3b}{2y}\right) \div \left(2-\dfrac{3b}{2y}\right)$.

21. $\dfrac{r^2-9s^2}{r^2-4s^2} \div \dfrac{r^2+rs-6s^2}{r^2-rs-6s^2}$.

22. $\left(\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}\right) \div \left(\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}\right)$.

[HINT. Change these expressions to fractions as in Ex. 19.]

23. $\left(x+1-\dfrac{30}{x}\right) \div \left(x+4-\dfrac{5}{x}\right)$.

24. $\left(\dfrac{s^2}{n^2}+\dfrac{2s}{n}+1\right) \div \left(1+\dfrac{s}{n}\right)$.

25. $\dfrac{(a-b)^2-25}{(a+b)^2-25} \div \dfrac{a-b-5}{a+b+5}$.

* 26. $\dfrac{a^3-1}{x^3+1} \div \dfrac{a-1}{x+1}$.

* 27. $\left(\dfrac{a^2+ab+b^2}{x^2-xy+y^2} \div \dfrac{a^3-b^3}{x^3+y^3}\right) \cdot \dfrac{a-b}{x+y}$.

For further exercises on this topic, see the review exercises, *p.* 161, *and Appendix, p.* 303.

EXERCISES — APPLIED PROBLEMS

The algebraic processes needed in the following exercises are simple. The principal difficulty is to *read* the problem carefully and intelligently.

1. In the figure are two weights, one weighing W_1 lb. (read W *sub* 1 pounds) and the other weighing W_2 lb. (read W *sub* 2 pounds). They are joined by a string which runs over a freely turning wheel at the top (the wheel having a groove cut in it for the string to run in). When the two weights thus tied together are left to themselves, the heavier one gradually descends (goes down), pulling up the lighter weight on the other side. It is shown in physics that during the motion the force T tending to break the string (called the *tension*) has the value given by the formula

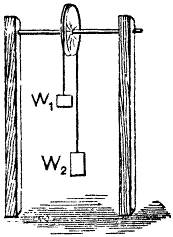


FIG. 38.

$$T = \frac{2\,W_1W_2}{W_1+W_2}\text{ lb.}$$

By means of this formula, answer each of the following questions:

(*a*) If $W_1=3$ lb. and $W_2=1\frac{1}{2}$lb. what is the tension in the string during the motion? 2 lb. *Ans.*

(*b*) If $W_1=W_2$, show that the tension becomes equal to either of the weights.

In this case there is no motion.

(*c*) If $W_1=5$ lb., what must W_2 be in order that the tension during the motion be 4 lb.?

(*d*) Show that if at any time the weight W_1 be increased by q lb., the tension T will be increased by

$$\frac{2\,qW_2^2}{(W_1+W_2)(W_1+W_2+q)}\text{ lb.}$$

2. The figure represents a beam (wood or steel or any other material) with its ends resting upon two level supports (called *abutments*), as is illustrated, for example, in the girder of a bridge. If a weight (usually called a *load*) of W lb. is suspended from the beam at the point P which is a feet from one abutment and b feet from the other, then the upward pressures (called *thrusts*) on the beam at its ends (where it rests on the abutments) are

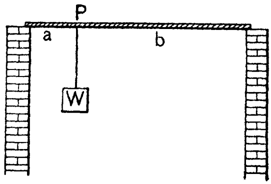


FIG. 39.

$$\frac{bW}{a+b} \text{ and } \frac{aW}{a+b},$$

the first of these being the thrust at the end which is distant a feet from P, and the second being the thrust at the end which is distant b feet from P.

Whence, show the following facts about such a beam:

(*a*) The two thrusts are equal when P is the middle point of the beam; that is, when $a=b$.

(*b*) The sum of the thrusts is in all cases precisely equal to the load W suspended.

3. When an automobile weighing 2 tons is $\frac{1}{4}$ the way across a bridge, how much of its weight is being supported by the nearer abutment; how much by the farther one? (Answer by the formulas in Ex. 2.)

4. A man writes: "We have a set of hay-scales, and sometimes we have to weigh wagons that are too long to go on them. Can we get the correct weight by weighing one end at a time and then adding the two weights?" (Answer by means of Ex. 2.)

103. Complex Fractions.** A fraction whose numerator or denominator (or both) contains fractions is called a ***complex fraction.

Thus

$$\frac{\frac{a+b}{c}}{x} \text{ and } \frac{\frac{z-a}{b}}{1+\frac{z}{b}}$$

are complex fractions. Since a fraction is an indicated division, it follows that the expression in the numerator of a complex fraction is to be divided by the expression in the denominator. A complex fraction can be reduced to a simple one by simplifying the numerator and the denominator and then performing the division indicated.

EXAMPLE 1. Simplify the complex fraction $\frac{\frac{a}{b}}{\frac{c}{d}}$.

SOLUTION. $$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}. \quad Ans.$$

EXAMPLE 2. Simplify $\frac{a}{x+\frac{y}{z}}$.

SOLUTION. $$\frac{a}{x+\frac{y}{z}}=\frac{a}{\frac{xz+y}{z}}=a\cdot\frac{z}{xz+y}=\frac{az}{xz+y}. \quad Ans.$$

EXAMPLE 3. Simplify $\frac{4+\frac{5a}{3b}}{\frac{a}{b}-3}$.

SOLUTION. This fraction may be simplified, as in Exs. 1 and 2; or we may simply multiply both terms of the fraction by $3b$ (which is the least common denominator of the simple fractions). Thus, we obtain

$$\frac{\left(4+\frac{5a}{3b}\right)\cdot 3b}{\left(\frac{a}{b}-3\right)\cdot 3b}=\frac{12b+5a}{3a-9b}. \quad Ans.$$

WRITTEN EXERCISES

Simplify the following.

1. $\dfrac{6}{\frac{3}{4}}$.

2. $\dfrac{\frac{1}{2}}{\frac{5}{6}}$.

3. $\dfrac{2\frac{1}{3}}{3\frac{3}{4}}$.

4. $\dfrac{\dfrac{1}{a}}{\dfrac{3b^2}{2a}}$.

5. $\dfrac{\dfrac{3a^2b}{4xy^3}}{\dfrac{ab^4}{2x^2y}}$.

6. $\dfrac{a+b}{\dfrac{1}{a+b}}$.

7. $\dfrac{a+\dfrac{y}{x}}{a-\dfrac{y}{x}}$.

8. $\dfrac{\dfrac{1}{x-y}}{\dfrac{1}{x+y}}$.

9. $\dfrac{\dfrac{a}{4}-\dfrac{b}{3}}{\dfrac{a}{2}-\dfrac{b}{6}}$.

10. $\dfrac{1+\dfrac{x-y}{x+y}}{1-\dfrac{x+y}{x-y}}$.

11. $\dfrac{b+\dfrac{a}{c}}{1-\dfrac{c}{b}}$.

12. $\dfrac{\dfrac{1}{x+4}}{1+\dfrac{1}{x-2}}$.

13. $\dfrac{\dfrac{a}{1+b}+1}{\dfrac{a}{1-b}-1}$.

14. $\dfrac{\dfrac{2}{a^2}+\dfrac{3}{a}-1}{2+\dfrac{1}{a}-\dfrac{1}{a^2}}$.

15. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}$.

16. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}}$.

17. $\dfrac{\dfrac{x^2-y^2}{z^2}}{\dfrac{x+y}{z}}$.

EXERCISES — REVIEW OF CHAPTER IX

1. What principle of fractions is used in reducing fractions to lowest terms?

Reduce to lowest terms.

2. $\dfrac{a^2-2a-15}{a^2+2a-35}$.

3. $\dfrac{4x^2-9}{4x^2-12x+9}$.

4. $\dfrac{x^3-3x^2-4x}{x^3-8x^2+16x}$.

5. $\dfrac{rs+ry-bs-by}{cs+cy-ks-ky}$.

6. $\dfrac{a(a+2b)^4(a^3+2a^2b+ab^2)}{b(a^2-4b^2)^2(a^5-2a^3b^2+ab^4)}$.

Since an integer may be regarded as a fraction with the denominator 1 (see Note, § 101) reduce each of the following expressions to a single fraction.

7. $a+\frac{a^2-ab}{a}$. **8.** $x-\frac{a-c-s}{d}$. **9.** $1+m+\frac{m^2}{1+m}$.

Perform the indicated additions and subtractions in the following exercises.

10. $\frac{10}{4x+24}+\frac{3}{6x+36}$.

11. $\frac{y}{x(x+y)}+\frac{1}{x-y}$.

12. $\frac{a-b}{a^2+ay}-\frac{a}{a^2-y^2}$.

13. $\frac{2}{b}-\frac{2a}{a-b}-\frac{2ab}{(a-b)^2}$.

14. $\frac{2}{a-1}+\frac{2}{a+1}-\frac{4a}{a^2-2a+1}$.

15. $\frac{2a+3}{a+3}+\frac{a-4}{a-5}-\frac{3a^2-8a-27}{a^2-2a-15}$.

16. $1+\frac{1}{1+x}+\frac{1}{1-x}$.

17. $\frac{1}{1-x}-1+\frac{1}{1+x}$.

18. $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}$.

19. $\frac{1}{a+b}-\frac{1}{a-b}+\frac{1}{a-\frac{b^2}{a}}$.

20. Changing the signs of the numerator and denominator of a fraction is the same as multiplying the numerator and denominator by -1.

Apply this to explain each of the following reductions:

$$\frac{4}{a^2-b^2}=\frac{-4}{b^2-a^2};\quad \frac{a}{3(a-b)}=\frac{-a}{3(b-a)}.$$

Perform the multiplication indicated in each of the following exercises.

21. $\left(x-\frac{y}{z}\right)\left(x+\frac{y}{z}\right)$.

22. $\left(ab+\frac{ab}{a-b}\right)\left(ab-\frac{ab}{a+b}\right)$.

23. $\frac{a^2-b^2}{x+y}\cdot\frac{x^2-y^2}{a+b}\cdot\frac{c^2}{x-y}$.

24. $\frac{x^2-5x-24}{x^2-5x+6}\cdot\frac{x^2-2x-8}{x^2-4x-21}\cdot\frac{x^2-9x+14}{x^2-6x-16}$.

25. $\dfrac{r^4-s^4}{(r-s)^2}\cdot\dfrac{r-s}{r^2+rs}\cdot\dfrac{r^2}{r^2+s^2}.$

26. $\dfrac{x(x-y)}{x^2+2\,xy+y^2}\cdot\dfrac{x(x+y)}{x^2-2\,xy+y^2}.$

27. $\left(1-\dfrac{a-1}{a^2+6\,a+5}\right)\left(1-\dfrac{2}{a^2+7\,a+12}\right).$

The ***reciprocal*** of a number is 1 divided by that number. Thus, the reciprocal of x is $1/x$; of ab is $1/ab$; of a/b is $1/(a/b)$, or b/a.

28. State the reciprocal of $5,\ \dfrac{2}{3},\ \dfrac{a}{2},\ \dfrac{2}{a},\ \dfrac{x+y}{2},\ \dfrac{a+b}{a-b}.$

29. Show that multiplying a/b by the reciprocal of a gives the same result as dividing a/b by a.

30. Show that the sum of any two numbers, as a and b, divided by their product, is equal to the sum of their reciprocals.

Perform the operations indicated in each of the following exercises.

31. $\dfrac{a^4-b^4}{a^2-2\,ab+b^2}\div\dfrac{a^2+b^2}{a^2-ab}.$

32. $\left(a+\dfrac{1}{b}\right)\div\left(a-\dfrac{1}{b}\right).$

33. $(6\,a+3)\div\dfrac{4\,a+2}{3\,a}.$

34. $\left(\dfrac{2\,a+b}{a+b}+1\right)\div\left(\dfrac{2\,a+b}{a+b}-1\right).$

35. $\dfrac{x^2-13\,x+22}{x^2-9\,x+8}\div\dfrac{x^2-5\,x+6}{x^2-6\,x-16}.$

36. $(r+s)\div\left(\dfrac{r^2-s^2}{1+y}\div\dfrac{r-s}{1-y^2}\right).$

37. $\left(\dfrac{y^2}{1-y^2}+\dfrac{2\,y}{1-y}\right)\div\dfrac{1-y}{y+1}.$

38. $\left(\dfrac{2\,a(a^2-r^2)}{5\,b(d^2-r^2)}\div\dfrac{a^2-ar}{bd+br}\right)\cdot\dfrac{a^2+2\,ar+r^2}{d^2-2\,dr+r^2}.$

39. $\dfrac{\dfrac{a}{b^2}}{1-\dfrac{a^2}{b^2}}.$

*** 40.** $\dfrac{1+\dfrac{1}{x^2}+\dfrac{1}{x^4}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}.$

*** 41.** $\left[\dfrac{1-\dfrac{b}{a}+\dfrac{b^2}{a^2}}{1+\dfrac{b}{a}+\dfrac{b^2}{a^2}}\cdot\dfrac{\dfrac{a^3}{b^3}-1}{\dfrac{a^3}{b^3}+1}\right]\div\dfrac{\left(\dfrac{1}{a}-\dfrac{1}{b}\right)^2}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2}.$

For further exercises on this chapter, see Appendix, pp. 301–303.

CHAPTER X

FRACTIONAL EQUATIONS

104. Fractional Equations. A *fractional equation* is one which contains fractional expressions.

105. Clearing of Fractions and Solving. Multiplying both members of an equation by such a number as will cancel each denominator is called ***clearing of fractions***.

Thus, multiplying both sides of the equation $x/2=6$ by 2, we get $x=12$, an equation cleared of fractions.

Multiplying both sides of the equation $\frac{x}{3}+\frac{1}{12}=\frac{1}{4}$ by 12, we get $4x+1=3$, an equation cleared of fractions.

EXAMPLE 1. Solve the fractional equation

$$\frac{2x}{3}+\frac{3x}{4}+\frac{5x}{6}=9.$$

SOLUTION. Multiplying both sides of the equation by 12, the L. C. M. of the denominators, we find,

$$8x+9x+10x=108.$$

Combining like terms,

$$27x=108.$$

Dividing by 27,

$$x=4. \quad Ans.$$

CHECK. Substituting 4 for x in the given equation, we have

$$\frac{8}{3}+\frac{12}{4}+\frac{20}{6}=9,$$

or

$$\frac{16+18+20}{6}=9,$$

which is a *true* statement.

EXAMPLE 2. Solve $\frac{x-1}{4}-\frac{x-2}{2}=\frac{x-3}{8}-\frac{3}{2}$.

SOLUTION. Multiplying both sides by 8, the L. C. M. of the denominators, we find

$$2x-2-4x+8=x-3-12. \quad \text{(Note the signs here carefully.)}$$

Transposing,

$$-x+2x-4x=-3-12+2-8.$$

Combining,

$$-3x=-21.$$

Dividing by -3,

$$x=7. \quad \textit{Ans.}$$

CHECK. Substituting 7 for x in the first equation gives

$$\frac{7-1}{4}-\frac{7-2}{2}=\frac{7-3}{8}-\frac{3}{2}, \quad \text{or } \frac{6}{4}-\frac{5}{2}=\frac{4}{8}-\frac{3}{2}, \quad \text{or } -1=-1,$$

which is a *true* statement.

NOTE. In the given equation of Example 2 the sign before the fraction $(x-2)/2$ is minus. The line between the numerator and the denominator has the same effect as a parenthesis around the numerator, for when the equation is cleared of fractions and this line is removed the sign of each term in the numerator is changed. See § 93.

ORAL EXERCISES

Solve each of the following equations by clearing of fractions.

1. $\frac{x}{2}=1.$
2. $\frac{y}{5}=1.$
3. $1=\frac{x}{3}.$
4. $1=\frac{5}{x}.$
5. $\frac{1}{x}=-1.$
6. $\frac{-2}{x}=1.$
7. $\frac{1}{x}=4.$
8. $\frac{1}{2x}=3.$
9. $\frac{1}{2x}=-1.$
10. $4=\frac{2}{x}.$
11. $\frac{1}{x}-2=0.$
12. $2-\frac{1}{x}=0.$
13. $4+\frac{1}{y}=0.$
14. $-\frac{1}{y}+3=0.$
15. $\frac{1}{r}+3=6.$

WRITTEN EXERCISES

Solve each of the following equations by first clearing of fractions. Check each answer for Exs. 1–15.

1. $\frac{3x}{4}-9=6.$ $x=20.$ *Ans.*

2. $\frac{x}{3}+5=\frac{3x}{4}.$

3. $\frac{x}{5}+2x=22.$

4. $4x-\frac{x}{3}=55.$

5. $\frac{x}{6}-\frac{10}{3}=-\frac{x}{9}.$

6. $\frac{2x}{4}-\frac{2}{3}=\frac{x}{3}+\frac{1}{3}.$

7. $\frac{2x}{3}-\frac{x}{2}-\frac{x}{5}=\frac{x}{6}-6.$

8. $x+\frac{x}{9}-\frac{x}{12}=31+\frac{x}{6}.$

9. $\frac{3x-1}{2}-\frac{2}{3}=\frac{x-3}{4}+\frac{6x+5}{6}.$

[HINT. See Example 2, § 105.]

10. $\frac{x-4}{4}+\frac{1}{6}=\frac{15-x}{3}.$

11. $\frac{x-3}{7}-\frac{x+2}{6}=4-\frac{x+5}{3}.$

[HINT. See Note, § 101.]

12. $\frac{x+3}{4}-\frac{x+4}{5}=16-\frac{x+1}{3}.$

13. $\frac{s-2}{3}-1=\frac{s+1}{6}-\frac{s+2}{7}.$

14. $\frac{r-2}{3}-\frac{r-3}{4}=6-\frac{r-1}{2}.$

15. $\frac{5y+1}{2}-\frac{17y-5}{3}=5-\frac{10y+14}{4}.$

16. $\frac{3x}{4}-\frac{4x-2}{5}=5-\frac{5x}{8}.$

17. $\frac{u+2}{6}-\frac{u+5}{3}=\frac{u-3}{7}-4.$

18. $\frac{3x+5}{5}-\frac{8x-60}{20}=8.$

19. $\frac{2x+11}{3}-9=\frac{3x-11}{11}$

20. $\frac{2s-1}{3}-1=7s-\frac{s+2}{2}-7.$

21. $\frac{x}{4}+\frac{x}{9}-5=0.$

22. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13.$

23. $\frac{2x}{11}-\frac{x-1}{6}=2.$

24. $\frac{r-1}{6}+10=\frac{3r+1}{2}-\frac{2r}{3}.$

25. $\frac{13-30x}{20}-\frac{9-80x}{6}-\frac{50x-4}{3}=0.$

26. $\frac{x-1}{4}-\frac{x-2}{3}=\frac{x-3}{2}-\frac{5}{12}.$

27. $\frac{1}{x}+\frac{1}{2x}=\frac{3}{8}.$

[HINT. The L. C. M. of the denominators is $8x$.]

28. $\frac{6}{5x}-\frac{3}{7x}=\frac{9}{70}.$

29. $\frac{4}{x}+1=\frac{6}{x}.$

30. $\frac{6}{x}+\frac{5}{x}=16\frac{1}{2}.$

31. $\frac{r-3}{2r}-\frac{5}{12}=0.$

32. $\frac{5}{r}+\frac{1}{5}-\frac{3}{r}=\frac{1}{3}.$

33. $\frac{1}{y}+\frac{2}{y}+\frac{3}{y}=6.$

34. $\frac{1}{y}-\frac{y+1}{y^2}=\frac{2}{3y}.$

35. $\frac{9r+5}{5r}-\frac{3r+1}{4r}-7=-\frac{5r-1}{8r}-\frac{21}{5}-\frac{5}{r}.$

Equations having denominators with several terms are solved in the same way as those having denominators with one term.

36. Solve the equation

$$\frac{3x+2}{x+1}+\frac{5}{x-1}-\frac{9}{x^2-1}=3.$$

Solution. Multiplying both sides by x^2-1, the L. C. M. of the denominators, gives

$$3x^2-x-2+5x+5-9=3x^2-3.$$

Canceling the $3x^2$ and transposing,

$$-x+5x=2-5+9-3.$$

Collecting like terms,

$$4x=3, \qquad \text{or, } x=\tfrac{3}{4}, \quad \textit{Ans.}$$

Check. Placing $x=\frac{3}{4}$ in the given equation, we obtain

$$\frac{3\cdot\frac{3}{4}+2}{\frac{3}{4}+1}+\frac{5}{\frac{3}{4}-1}-\frac{9}{(\frac{3}{4})^2-1}=3,$$

or,

$$\frac{\frac{17}{4}}{\frac{7}{4}}+\frac{5}{-\frac{1}{4}}-\frac{9}{-\frac{7}{16}}=3,$$

or,

$$\frac{17}{7}-20+\frac{144}{7}=3, \qquad \text{or, } \frac{21}{7}=3,$$

which is seen to be correct.

37. $\dfrac{4}{x+3}=2.$

38. $\dfrac{1}{x+1}-\dfrac{2}{3(x+1)}=\dfrac{1}{12}.$

39. $\dfrac{3}{5(x-1)}-\dfrac{4}{3x-3}=-\dfrac{2}{15}.$

40. $\dfrac{1}{4-6y}-\dfrac{1}{2-3y}+\dfrac{1}{6}=0.$

41. $\dfrac{r-4}{r-7}+\dfrac{r-9}{r-12}=2.$

42. $\dfrac{2x+5}{3x-5}-1=0.$

43. $\dfrac{1}{r^2-r-2}+\dfrac{r+1}{r-2}=\dfrac{r-2}{r+1}.$

44. $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}=\dfrac{20}{x^2-4}.$

45. $\dfrac{x+1}{x-1}-\dfrac{x+3}{x+3}=\dfrac{8}{x}.$

46. $\dfrac{t+2}{t+3}=1-\dfrac{2}{t+5}.$

47. $-\dfrac{3}{r-2}=\dfrac{1}{r+2}-\dfrac{2}{2-r}.$

[HINT. Write $-\dfrac{2}{2-r}$ in the form $+\dfrac{2}{r-2}$. (See § 93.)]

48. $\dfrac{r+3}{r+2}-\dfrac{r+2}{r+3}=\dfrac{6}{r^2+5r+6}.$

49. $\dfrac{x+2}{x-4}-\dfrac{x-3}{x-8}=\dfrac{3x+8}{x^2-12x+32}.$

EXERCISES — APPLIED PROBLEMS

1. Divide 64 into two parts such that if one part is divided by 5 and the other by 7, the sum of the quotients shall be 10.

SOLUTION. Let x be one part. Then, $64-x=$ the other part. From the statement of the problem, we have

$$\frac{x}{7}+\frac{64-x}{5}=10.$$

Clearing of fractions,

$$5x+448-7x=350.$$

Transposing,

$$5x-7x=350-448.$$

Combining,

$$-2x=-98.$$

Dividing by -2,

$$x=49, \textit{ one part.}$$

$$64-49=15, \textit{ other part.} \quad \textit{Ans.}$$

2. If 18 be subtracted from a certain number, three fourths of the remainder is 30. What is the number?

3. The difference between $\frac{1}{2}$ and $\frac{1}{3}$ of a certain number is 16. What is the number?

4. Divide 78 into two parts whose quotient is $\frac{5}{8}$.

5. Divide 96 into two parts such that $\frac{3}{4}$ of the greater shall exceed $\frac{3}{4}$ of the smaller by 24.

6. The sum of two numbers is 179. If the larger is divided by the smaller, the quotient is 3 and the remainder 11. Find the numbers.

[HINT. Let x=the larger number; then, $179-x$=the smaller number. The equation becomes

$$\frac{x}{179-x}=3+\frac{11}{179-x}. \quad \text{(Why ?)]}$$

7. Separate 286 into two parts such that the greater divided by the smaller gives a quotient 2 and a remainder 28.

8. One number is 222 greater than a certain other one. If the larger is divided by the smaller, the quotient is 3 and the remainder is 18. What are the numbers?

9. What number added to the numerator and denominator of $\frac{3}{5}$ will change the value of the fraction to $\frac{6}{7}$?

$\left[\text{HINT. The equation is } \frac{3+x}{5+x}=\frac{6}{7}. \text{ Explain.}\right]$

10. What number added to the numerator and denominator of $\frac{11}{25}$ will change the value of the fraction to $\frac{13}{20}$?

11. What number added to the numerator and denominator of $\frac{21}{23}$ will change the value to $\frac{5}{6}$?

12. What number added to the numerator and subtracted from the denominator of $\frac{47}{49}$ will change the value to $\frac{22}{23}$?

13. Two sevenths of a certain angle is 23° less than $\frac{8}{5}$ of its complement. How large is the angle (in degrees)?

[HINT. The complement of an angle is the difference between 90° and that angle, *i.e.* the complement of any angle x is $90°-x$.]

14. If $\frac{5}{27}$ of the supplement of a certain angle equals $\frac{5}{9}$ of its complement, how large (in degrees) is the angle?

[HINT. The supplement of an angle is the difference between 180° and that angle, *i.e.* the supplement of any angle x is $180°-x$.]

15. The complement of a certain angle is $\frac{5}{23}$ of its supplement; find the angle.

16. A 14-foot piece of timber must be cut into two parts one of which is $\frac{3}{7}$ the length of the other. Find the lengths of the two parts.

17. A teacher asked a class to divide one half a certain number by 4 and the other half by 12. Instead of doing so, one member of the class divided the entire number by half the sum of the two divisors. His answer was too small by 4. What was the number?

18. A man started on a journey with a certain sum of money. He spent $\frac{1}{6}$ for car fare, $\frac{1}{2}$ of the remainder for hotel bills. When he returned home he found that he had \$25.00. How much did he start with?

19. A man lost $\frac{2}{3}$ of his money in speculation, lent $\frac{1}{6}$ of the remainder to a friend, and spent the rest for a home, paying \$5460 for it. What was his original capital?

20. The difference between two numbers is 22 and $\frac{2}{3}$ of the greater exceeds $\frac{1}{2}$ the smaller by 19. Find the numbers.

21. A tank can be filled by one pipe in 10 hours, and by another pipe in 15 hours. How long will it take to fill the tank if *both* pipes are open together?

SOLUTION. Let $x=$ the number of hours.

Then $\frac{1}{x}=$ the part of the tank both can fill in one hour.

$\frac{1}{10}=$ the part of the tank the first pipe can fill in one hour.

$\frac{1}{15}=$ the part of the tank the second pipe can fill in one hour.

Therefore,

$$\frac{1}{x}=\frac{1}{10}+\frac{1}{15}.$$

Clearing of fractions,

$$30=3\,x+2\,x.$$

Transposing and combining,

$$5\,x=30.$$

Therefore, $x=6$ hours, the required time. *Ans.*

22. How long will it take two pipes to fill a tank if one can fill it in 5 hours and the other can fill it in 12 hours?

23. Two pipes are connected with a tank. The large one can fill it in 7 hours; the small one can empty it in 9 hours. How long will it take to fill the tank if both pipes are open?

24. A plows a field of corn in 4 days, and B plows it in 5 days. How long will it take to plow the field if they work together?

[HINT. If x = the number of hours it will take both, we get the equation $\frac{1}{x}=\frac{1}{4}+\frac{1}{5}$. Compare Ex. 21.]

25. A does a piece of work in 4 days, B in 6 days, and C in 8 days. How long will it take them working together?

26. A can do a piece of work in 16 hours, and B can do it in 20 hours. If A works for 10 hours, how many hours must B work to finish?

27. A brick mason can build a wall in 10 days; if another mason helps him, they can build it in $3\frac{1}{3}$ days. How long will it take the second mason alone?

28. It takes 5 hours for a railroad tank to be filled with water. Locomotives draw out and use a tank full every 8 hours. How long will it take to pump the tank full?

29. A's age is two fifths of B's; ten years ago one fourth of B's age equaled A's. What is the age of each?

30. A son's age is one third that of his father's. 12 years ago he was $\frac{1}{8}$ as old as his father. How old is each now?

31. Five gallons of 80% alcohol are to be made 50% alcohol. How many gallons of water must be added?

[HINT. The equation is $.50(x+5)=5\times.80$. Explain.]

32. How much water must be added to 65 lb. of a 10% solution of salt to make it an 8% solution?

33. 18% of the weight of wheat is lost in grinding it into flour. How many bushels of wheat, 60 lb. each, must be used to make 984 lb. of flour?

34. Separate 169 into two parts such that if one part is divided by 21 and the other by 14, the sum of the quotients is 11.

35. If a certain number is subtracted from the numerator and added to the denominator of the fraction $\frac{7}{8}$, the fraction is diminished by $1\frac{1}{2}$. What is the number?

36. The fare from St. Louis to Chicago and back is $15.00. A special rate was made in connection with a convention so that I made the trip, paid my expenses which were equal to $2\frac{1}{2}$ times my fare, and spent all together only $2.50 more than the regular fare. What was the special rate?

37. Democritus lived $\frac{1}{4}$ of his life as a boy, $\frac{1}{5}$ as a youth, $\frac{1}{3}$ as a man, and spent 13 years in his dotage. How old was he when he died? (From a collection of problems by Metrodorius.)

38. An aviator flew 75 miles and back in 4 hours. His rate when returning was 40 miles per hour. What was his rate going?

[HINT. If one travels s miles at the rate of r miles an hour, the time t occupied is given by the formula $t = s/r$ hours.]

39. An aviator made a trip of 95 miles. After flying 40 miles he increased his speed 15 miles an hour and made the remaining distance in the same time as it took him to fly the first 40 miles. What was his rate during the first 40 miles of the trip?

For further exercises on this topic, see Appendix, p. 303.

106. Literal Equations. Equations in which some, or all, of the known numbers are represented by letters are called ***literal equations***.

Thus, $x-a=b$, $\frac{b}{x}+a=c$, $2\,ax-b=3$, are literal equations.

Here the first letters of the alphabet, a, b, c, etc., represent (as usual) the *known* numbers, while x represents the *unknown* number.

Literal equations are solved by the same methods as the equations in § 105.

EXAMPLE 1. Solve for x in the equation

$$ax-b=2\,b-2\,ax.$$

SOLUTION. Transposing, we find

$$ax+2\,ax=2\,b+b.$$

Combining like terms,

$$3\,ax=3\,b.$$

Dividing by $3\,a$,

$$x=\frac{b}{a}. \quad Ans.$$

CHECK. Substituting $\frac{b}{a}$ for x in the given equation gives

$$a\left(\frac{b}{a}\right)-b=2\,b-2\,a\left(\frac{b}{a}\right).$$

$$b-b=2\,b-2\,b.$$

$$0=0.$$

NOTE. Observe carefully that the solution $\frac{b}{a}$ is in terms of the *known* letters; that is, it contains only the a and the b. This must be true of the solution of every literal equation.

EXAMPLE 2. Solve for x in the equation $ax=bx+7\,c$.

SOLUTION. Transposing, we find

$$ax-bx=7\,c.$$

Combining,

$$(a-b)x=7\,c.$$

Dividing by $a-b$,

$$x=\frac{7c}{a-b}. \quad \textit{Ans.} \quad \text{(See Note on p. 174.)}$$

CHECK. Substituting the answer for x in the given equation gives

$$a\left(\frac{7c}{a-b}\right)=b\left(\frac{7c}{a-b}\right)+7c,$$

or,
$$7ac=7bc+7ac-7bc,$$

or,
$$7ac-7bc=7ac-7bc,$$

which is seen to be true.

ORAL EXERCISES

Solve for x in each of the following equations.

1. $x-a=b.$
2. $ax=cd.$
3. $ax-1=c.$
4. $ax+bx=c.$
5. $2+cx=r.$
6. $\frac{x}{a}=b.$
7. $\frac{x}{a}=\frac{2}{c}.$
8. $\frac{x}{a+b}=a+b.$
9. $r^2x=r-4.$
10. $ac^2bx=a^2c^3b^2.$
11. $(a+b)x=c.$
12. $(a-c)x=(a-c)d.$
13. $ax+cx=5.$
14. $ax+cx=0.$
15. $\frac{a}{x-a}=2.$

WRITTEN EXERCISES

In the following equations the first letters of the alphabet are the *known* letters. Solve for x. Remember that in each case it is necessary to get the value of x *in terms of the known letters*. (See Note, § 106.)

1. $3x+b=x-3b.$
2. $3(a-x)=12d.$
3. $(a+x)^2=a^2+x^2.$

[HINT. Use Formula V, § 59, to simplify the first member.]

4. $4(3b-x)=3(2b+x)$.

5. $(a+b)(x+b)=(a+b)2c$.

6. $(x-a)(x-b)=x(x+c)$.

7. $(x-b)(a-c)=(b-c)(x-a)$.

8. $\frac{1}{1+x}=a$.

9. $\frac{a}{x}+\frac{b}{x}=2$.

10. $\frac{x}{a}+\frac{x}{b}=2$.

11. $\frac{x}{a}+b=\frac{x}{b}+a$.

12. $\frac{a}{cx}+\frac{b}{dx}=e$.

13. $\frac{x-c}{x}-\frac{2x}{x-c}=-1$.

14. $\frac{2x+3}{3a}-\frac{4x+2}{a}=\frac{1}{3}$.

[HINT. See § 92.]

15. $\frac{2x}{ab}+\frac{3x}{bc}+\frac{x}{ac}=1$.

16. $\frac{x-c}{c}+a=x-1$.

17. $\frac{x+b}{4}-\frac{4}{x+b}=\frac{x-b}{4}$.

18. $\frac{x-a}{x-b}=\frac{x-2}{x-4}$.

19. $\frac{x-b}{x-3}+\frac{x-c}{x+2}=2$.

20. $\frac{2x+k}{r^2-k^2}=\frac{r}{r+k}-1$.

21. $\frac{x+a}{x+b}=\left(\frac{2x+a}{2x+b}\right)^2$.

$\left[\text{HINT.}\quad \left(\frac{2x+a}{2x+b}\right)^2=\left(\frac{2x+a}{2x+b}\right)\left(\frac{2x+a}{2x+b}\right)=\frac{(2x+a)^2}{(2x+b)^2}.\quad \text{See § 100.}\right]$

22. $\frac{ax}{b}-\frac{c}{d}=\frac{b}{a}-\frac{dx}{c}$.

23. $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$.

24. $\frac{2b}{a}+\frac{c}{x}=\frac{b}{2-x}-\frac{2bx}{a(2-x)}$.

25. $a(b+x)-(b+x)(a-x)=x^2+\frac{ac^2}{b}$.

107. Formulas. If a person travels for 10 hours : rate of 15 miles an hour, we know from arithmetic tl.. total distance he goes will be $15 \times 10 = 150$ miles. St in general (algebraic) language, we can say in the same that if a person travels t hours at the rate of r miles an l the distance s which he will go is given by the for (called the law of uniform motion)

$$s = rt.$$

This is a literal equation which expresses the value o terms of r and t. If we wish, we can solve this equ for t, giving $t = \frac{s}{r}$, and what we now have is t expres terms of s and r. Or, we can solve the original equati r, which gives $r = \frac{s}{t}$ and this expresses r in terms of

This illustrates the very important fact that in r branches of knowledge, especially in engineering, g physics, and the like, there are *general* laws whic themselves in mathematical formulas. Such forı really nothing but literal equations in which twc letters appear, and it is often desirable to solve tions for some one letter in order to find its va' of the others.

EXERCISES — APPLIED PROBLEMS

1. The area A of a rectangle whose dimensi and breadth) are a and b is given by the form Solve this for a. Also, solve for b. In each case letters your answer is in terms of.

2. The formula for the area A of a triangle w (altitude) is h and whose base is a is $A = \frac{1}{2}$

Solve for a and solve for h, and state in e letters your answer is in terms of.

N

Solve for r in the formula $C=2\pi r$. (This is the formula ɜ circumference of a circle in terms of its radius. See 22, p. 22.)

Solve for B in the formula $A=\frac{1}{2}h(B+b)$. (This is formula for the area of a trapezoid whose bases are B b and whose altitude is h. See Ex. 28, p. 88.)

The interest I which a principal of p dollars will yield years at r per cent is determined by the formula $rt/100$. Solve this for r and use your result to answer ollowing question: What rate of interest is necessary ler that \$50 may yield \$6 interest in two years' time?

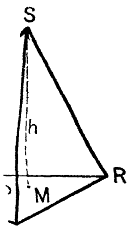
ɹUTION. Solving for r gives

$$r=\frac{100\,I}{pt}.$$

have only to see what this becomes when we put $I=6$, d $t=2$. The result is

$$r=\frac{100\times 6}{50\times 2}=6\%. \quad Ans.$$

ng the interest formula of Ex. 5, solve it for p and · result to answer the following question: How ʳincipal must be invested at 5% in order that it ıelded \$90 in interest by the end of three years?



40.

7. It is shown in solid geometry that the volume V of any pyramid is equal to one third the product of the area B of its base multiplied by the height h. That is, we have the formula $V=\frac{1}{3}Bh$. Solve this for h and use your answer to find how high a pyramid must be to contain one cubic foot, provided its base contains t. Answer the same when the base contains ɜ inches.

8. The formula for converting degrees Fahrenheit to degrees Centigrade is

$$C = \tfrac{5}{9}(F - 32).$$

Solve for F and use your result to answer the following questions:

(*a*) How many degrees Fahrenheit correspond to 0° Centigrade?

(*b*) How many degrees Fahrenheit correspond to 100° Centigrade?

9. The figure represents a body (of any material) submerged in a liquid. If the base of the body contains A square feet and is everywhere at a depth of h feet from the top of the liquid, then the upward pressure P on the base (due to the liquid) is given by the formula $P = wAh$, where w represents the weight of 1 cubic foot of the liquid.

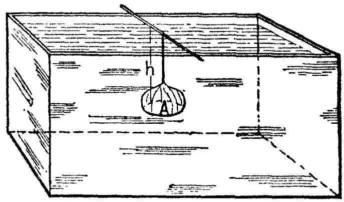


FIG. 41.

Find the pressure per square foot of surface near the bottom of a standpipe in which the water is 40 feet high, it being given that fresh water weighs 62.5 lb. per cubic foot.

10. Find by the formula in Ex. 9 the pressure per square foot at the bottom of the ocean at a depth of 3000 feet, it being given that sea water weighs 64 pounds per cubic foot.

11. The bottom of a rectangular cistern is 6 feet square. For what depth of water will the total pressure on the bottom be 18 tons? [HINT. Solve the formula of Ex. 9 for h.]

12. How deep in the ocean can a diver go without danger in a diving armor that can safely sustain a pressure of no more than 140 pounds per square inch?

CHAPTER XI

RATIO AND PROPORTION

108. Ratio. The quotient of one number divided by another of the same kind is called their ***ratio.***

Thus, the ratio of 12 inches to 6 inches is the fraction $\frac{12}{6}$, or $\frac{2}{1}$. The ratio of 2 feet to 3 feet is the fraction $\frac{2}{3}$. The ratio of 10 cents to $1 is $\frac{10}{100}$, or $\frac{1}{10}$. Note that in every case a ratio is simply a fraction of the kind studied in arithmetic.

The first number, or dividend, is called the ***antecedent;*** the second number, or divisor, is called the ***consequent.***

Thus, in the ratio $\frac{6}{7}$, the antecedent is 6 and the consequent is 7.

EXERCISES

1. What is the ratio of 5 quarts to 8 quarts? of 5 quarts to 10 quarts?

2. What is the ratio of 18 inches to 3 inches? of 18 inches to 1 foot?

3. What is the ratio of a foot to a yard? of a yard to an inch?

4. A stick was divided into two parts one of which contained 2 units and the other 7 units. What was the ratio of the two parts?

5. State (as a fraction in simplest form) the value of each of the following ratios.

(*a*) 8 to 12. (*c*) 72 to 36. (*e*) 30 to 10. (*g*) 60 to 15.
(*b*) 15 to 8. (*d*) 80 to 75. (*f*) 18 to 24. (*h*) 7.5 to 2.5

6. Find the value of each of the following ratios.

(*a*) $\frac{1}{2}$ to $\frac{1}{3}$. (*b*) $\frac{8}{9}$ to $\frac{9}{8}$. (*c*) $1\frac{1}{2}$ to $2\frac{2}{3}$.

7. Give (as a fraction) the simplest form for each of the following ratios.

(*a*) $2\,a$ to $2\,b$. (*b*) $3\,a^2$ to $6\,b^2$. (*c*) a^2-b^2 to $a+b$.

8. State which is the antecedent and which the consequent in each of the parts of Exs. 5, 6, and 7.

9. What is the ratio of one side of a square to any other side?

10. What is the ratio of the circumference of any circle to its radius?

SOLUTION. Call the radius r. Then the circumference (see Ex. 22, p. 22) will be $2\,\pi r$. Therefore, the ratio of the circumference to the radius will be $2\,\pi r/r$, or $2\,\pi$. *Ans.*

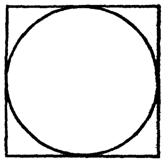


FIG. 42.

11. The figure shows a circle surrounded by a square which it just touches on all four sides. Find the ratio of the area of the circle to that of the square.

[HINT. Let r be the radius. Then a side of the square will be $2\,r$. Now proceed as in the solution of Ex. 10, using Ex. 25, p. 22, and remembering that the area of the square here will be $(2\,r)^2$, or $4\,r^2$.]

The circle in Fig. 42 is said to be *inscribed* in the square; the square is *circumscribed* about the circle.

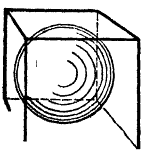


FIG. 43.

12. Find the ratio of the volume of any cube to that of the sphere that will just fit inside it.

[HINT. Let r be the radius of the sphere and use Ex. 28, p. 23.]

The sphere of Fig. 43 is said to be *inscribed* in the cube.

13. It is shown in geometry that the volume of a right circular cylinder is equal to the area of its base multiplied by its height. By means of this result show that when a sphere is completely surrounded by a cylinder in the manner shown in the figure (that is, the cylinder and sphere just touching each other above and below and on the side) then the ratio of the volume of the cylinder to the volume of the sphere is simply $\frac{3}{2}$.

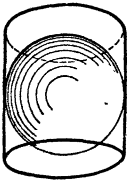


FIG. 44.

[HINT. Let r represent the radius of the sphere. Then the radius of the base of the cylinder will be r, and the height will be $2r$.]

14. Show that the circumferences of any two circles have the same ratio as their radii.

[HINT. Let R be one radius and r the other.]

15. Show that the areas of any two circles have the same ratio as the squares of their radii.

16. In the figure are two circles, each surrounded by a square which it just touches on all four sides. Show that

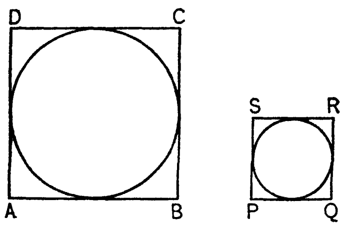


FIG. 45.

the areas of the two circles have the same ratio as the areas of their surrounding squares.

17. What is the ratio of the areas of two squares if the side of one is double the side of the other? Answer the same question for two circles if the radius of one is double that of the other.

18. What is the ratio of the volumes of two cubes if the edge of the one is double that of the other? Answer the same question for two spheres if the radius of the one is double that of the other.

19. Mr. A's automobile travels at the rate of 25 miles an hour, while Mr. B's travels at 20 miles an hour. What is the ratio of the time it will take A to make any given journey as compared to the time it will take B?

109. Proportion. A ***proportion*** is an expression of equality between two ratios, or fractions.

For example, since $\frac{2}{3}$ is the same as $\frac{4}{6}$, we have the proportion $\frac{2}{3}=\frac{4}{6}$. Likewise, we may write $\frac{1}{3}=\frac{2}{6}$, and $\frac{7}{5}=\frac{21}{15}$, and $-\dfrac{1}{3}=\dfrac{2}{-6}$; hence these are all true proportions. But $\frac{1}{3}=\frac{1}{2}$ is *not* a true proportion since the two fractions here are *unequal.*

Every proportion is thus seen to be an equality of the form $a/b=c/d$ where a, b, c and d stand for numbers. These four numbers are called the ***terms*** of the proportion. The first and fourth numbers (that is, a and d) are called the ***extremes***, while the second and third (b and c) are called the ***means***.

Besides writing a proportion in the form $a/b=c/d$, it may be written in the form $a:b=c:d$, or also in the form $a:b::c:d$. In all cases it is read "*a is to b as c is to d*," and it means that the *fraction a/b is equal to the fraction c/d.*

NOTE. Every proportion is thus a fractional equation of the kind studied in Chapter X.

EXERCISES

1. Using the language of proportion, read each of the following proportions.

(*a*) $\frac{5}{6}=\frac{10}{12}$.

(*b*) $3:4=6:8$.

(*c*) $-1:2::2:-4$.

(*d*) $\frac{1}{2}:\frac{1}{4}::2:1$.

2. State what are the extremes and what the means in each part of Ex. 1.

3. State any proportions that you can make out of the following four quantities: 2 inches, 8 inches, 4 inches, 16 inches.

[HINT. 2 inches is to 8 inches as . . .]

4. State any proportions that you can make out of the following four quantities: 1 inch, 3 inches, 1 foot, 1 yard.

[HINT. First express all the quantities in inches.]

5. State any proportions you can make out of the following four quantities: 1 pint, 1 quart, 1 gallon, 2 gallons.

6. Proceed as in Ex. 5 for the following quantities: 1 second, 1 minute, half an hour, a day and a half.

7. Proceed as in Ex. 5 for the following quantities: 1 cent, 1 dollar, 1 centimeter, 1 meter.

[HINT. Compare money ratio with distance ratio.]

8. Proceed as in Ex. 5 for the following quantities: 8 ounces, 1 pound, 1 pint, 1 quart.

9. Proceed as in Ex. 5 for the following quantities: 25 miles an hour, 30 miles an hour, 10 gallons of gasoline, 12 gallons of gasoline.

For further exercises on this topic, see Appendix, p. 306.

110. Algebraic Proportions. If we consider the algebraic fraction $(a^2b)/(ab^2)$ we see (upon dividing both numerator and denominator by ab) that it reduces to a/b. In other words, we have

$$\frac{a^2b}{ab^2}=\frac{a}{b}.$$

This is an example of an *algebraic proportion.* Similarly

$$\frac{2\,x^2y}{4\,xyz}=\frac{x}{2\,z}$$

is an algebraic proportion, and it may be written also in the form

$$2\,x^2y : 4\,xyz = x : 2\,z.$$

Likewise, since

$$\frac{a^2-b^2}{a-b}=a+b=\frac{a+b}{1},$$

we have

$$(a^2-b^2):(a-b)=(a+b):1.$$

111. Principle. Let $a/b=c/d$ be any proportion. By multiplying both sides of this equality by bd (Axiom III, § 9) we obtain

$$\frac{a}{\cancel{b}}\times\cancel{b}d=\frac{c}{\cancel{d}}\times b\cancel{d},\text{ or } ad=bc.$$

This result may be stated in words as follows:

PRINCIPLE. *In any proportion the product of the means* (see § 109) *equals the product of the extremes.*

This principle is often useful in testing the correctness of a proportion.

Thus, $6:9=14:21$ is a true proportion because the product of the means, which is 9×14, is equal to the product of the extremes, which is 6×21; but $6:9=8:15$ is *not* a true proportion because 9×8 is *not* equal to 6×15. Similarly $x^3:x^2y=x:y$ because $x^2y\cdot x=x^3\cdot y$.

EXERCISES

By means of the principle in § 111 test the correctness of each of the following proportions.

1. $\frac{8}{9}=\frac{40}{45}$.
2. $6:7=18:21$.
3. $5:-1=10:-2$.
4. $\frac{1}{2}:\frac{1}{4}=10:5$.
5. $19:21=42:105$.
6. $18:19=93:94$.
7. $-4:-5::16:20$.
8. $\dfrac{2.5}{1.2}=\dfrac{75}{36}$.
9. $6\,xy:13\,x^2=42\,y^2:91\,xy$.
10. $(x^2-y^2):(2\,x+2\,y)::(2\,x-2\,y):4$.
11. $(a^2-b^2):(a+b)^2::(a-b):(a+b)$.

[HINT. See Formula V, p. 101.]

12. $\dfrac{x^2+10\,x+16}{x^2+4\,x-32}=\dfrac{x+2}{x-4}$.

By means of the principle in § 111 find the value which x must have in each of the following proportions.

13. $\dfrac{x}{21}=\dfrac{1}{3}$.

[HINT. We must have $12\cdot 1=x\cdot 3$.]

14. $\dfrac{4}{6}=\dfrac{x}{10}$.
15. $15:x=12:8$.
16. $9:15=18:x$.
17. $\dfrac{25}{32}=\dfrac{8}{x}$.
18. $x:31=33:68$.
19. $\dfrac{21}{26}=\dfrac{x}{13}$.
20. $14:x=28:21$.
21. $(x-5):4::2:3$.

[HINT. We must have $4\cdot 2=(x-5)\cdot 3$.]

22. $x:(x-6)=9:14$.
23. $\dfrac{x-3}{x-4}=\dfrac{5}{6}$.
24. $\dfrac{x}{20}=\dfrac{5\,\pi}{8\,\pi}$.
25. What number bears the same ratio to 2 as 8 does to 3?

[HINT. Let x represent the unknown number and form a proportion. Solve for x.]

26. What number bears the same ratio to 7 as 2 does to 3?

27. Divide 35 into two parts whose ratio shall be $\frac{3}{4}$.

[HINT. Let x be one part. Then $35-x$ will be the other part.]

28. Divide 25 into two parts such that the greater increased by 1 is to the lesser decreased by 1 as 4 is to 1.

29. Two men divide \$6300 between them so that the parts are to each other in the ratio 3 : 4. How much does each receive?

30. A man's income from two investments is \$850. The two investments bear interests which are in the ratio of 6 to 8. What income does he receive from each?

31. Concrete for sidewalks is a mixture made of two parts of sand to one part of cement. How much of each is required to make a walk containing 500 cubic feet?

32. Find the number which, when added to each of the numbers 1, 2, 4, and 7, will give four numbers in proportion.

33. Prove that no four consecutive integers, as n, $n+1$, $n+2$, $n+3$, can form a proportion.

34. A bubble of air of volume v units when rising from a depth of d feet below the surface of the water gradually expands until it reaches a volume of V units at the surface such that

$$\frac{V}{v}=\frac{d+34}{34}.$$

Whence, find the volume at the surface of a spherical bubble which starts at a depth of 100 feet with a radius of 1 inch. [HINT. See Ex. 28, p. 23.]

35. Solve for d in the formula of Ex. 34 (see § 107) and use your result to answer the following question: From what distance below the surface must a bubble rise in order that its volume may increase from 3 cu. in. to 20 cu. in.?

112. The Lever. If a 2-pound weight be attached to one end of a yardstick and a 1-pound weight to the other end, and the whole be then exactly balanced, as shown in Figure 46, we have an example of a ***lever***. The point (pivot) around which the balance takes place is called the ***fulcrum***.

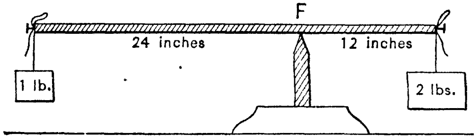


FIG. 46.

If this experiment be tried (and it easily can be at home or in the classroom) it will be found that when the balance is *exact*, the fulcrum is just 24 inches from one end of the stick, and 12 inches from the other end. Thus the ratio of these two distances is $\frac{24}{12}$, or $\frac{2}{1}$, which is therefore just the same as the ratio of the two *weights* to each other.

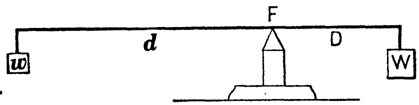


FIG. 47

This experiment illustrates a general law as follows:

LAW OF THE LEVER. *If two weights W and w are balanced at the ends of any uniform bar at distances D and d respectively from the fulcrum* (Fig. 47), *we have*

$$\frac{W}{w}=\frac{d}{D}, \text{ or } WD=wd.$$

EXERCISES — THE LEVER

1. In the figure above suppose $W=4$ pounds, $w=2$ pounds, and $D=6$ inches. What must be the value of d?

[HINT. By the law stated above, we have $4/2 = d/6$. Solve for d.]

2. Fill in each of the following question marks (?) in such a way that the balance will be perfect in the above figure.

(*a*) $W=9$ pounds, $w=3$ pounds, $d=1$ inch, $D=?$
(*b*) $W=8$ ounces, $w=4$ ounces, $d=?$, $D=1$ foot.
(*c*) $W=?$, $w=1\frac{1}{2}$ pounds, $d=\frac{1}{2}$ foot, $D=10$ inches.
(*d*) $W=3$ ounces, $w=1$ pound, $d=12$ centimeters, $D=?$

3. Where must we place the fulcrum under a 12-foot plank in order that a 56-pound boy at one end may balance a 112-pound boy at the other end?

[HINT. Let x be the distance from one end. Then the distance from the other end will be $12-x$.]

4. Two boys balance at seesaw on a 12-foot plank. The fulcrum is 5 feet from the heavier boy, who weighs 105 pounds. How much does the other boy weigh?

5. Sometimes, instead of having two weights balanced, we have a single weight balanced by a force, or, as it is usually called, a *power*. This may happen in several ways as indicated by the following figures.

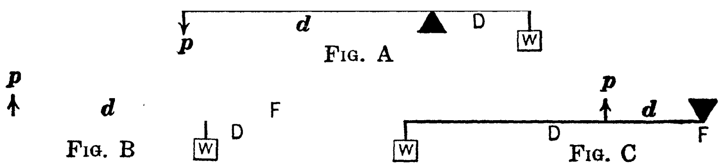


FIG. 48.

Note that in the last two figures the fulcrum is at one *end* of the bar.

In all these cases, if we let W represent the weight, p the power, D the distance of the weight from the fulcrum, and d the distance of the power from the fulcrum, we have

$$\frac{W}{p}=\frac{d}{D}, \text{ or } WD=pd.$$

This is called *the general law of the lever*.

By means of this law, answer the following question. If the fulcrum of a 5-foot crowbar is placed 1 foot from the end, what weight can be lifted by a man weighing 160 pounds?

[HINT. Here we have Fig. 48A with $W=$? $p=160$ lb., $D=1$ ft., $d=4$ ft.]

6. Figure 49 represents a simple form of pump. Suppose that the pump-handle AF is $1\frac{1}{2}$ ft. long, while the piston-arm FC is 5 inches long. What will be the upward

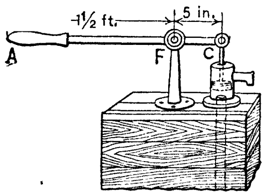


FIG. 49.

pull on the piston at C when there is a downward pressure of 10 pounds at A?

7. Figure 50 represents another common form of pump. With AF and FC measuring the same as in Ex. 6, what will

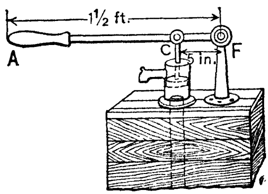


FIG. 50.

be the upward pull at C when there is a 10-pound upward pull at A?

[HINT. See Ex. 5, Fig. 48B.]

8. The drum (on which the string winds) of a windlass has a radius of 3 inches, and is turned by spokes 2 feet long. If we wish to raise a weight of 100 pounds, what force must we exert at the end of a single spoke?

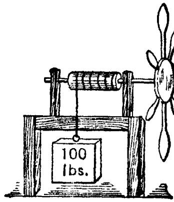


FIG. 51.

9. Figure 52 illustrates the ordinary form of safety valve on a steam engine boiler, consisting of a bar of iron hinged at one end F and weighted at the other end, and resting at some point between on a piston which sits upon the steam. If the area of the end of the piston (where it rests on the steam) is 16 square inches, what weight W must be hung on the end

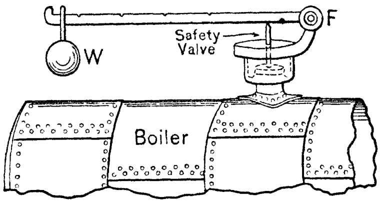


FIG. 52.

so that when the steam pressure p has risen to 100 pounds per square inch the valve will open, if W is $1\frac{1}{12}$ as far from F as from the piston rod.

10. In any of the levers illustrated in Ex. 5, the distance D from the weight to the fulcrum is called the *weight arm,* while the distance d from the power to the fulcrum is called the *power arm.* Show that the general law of the lever (see Ex. 5) may be stated in the following form, which is the way it is usually remembered by engineers: *The weight times the weight arm equals the power times the power arm.*

113. Gear Wheels. In the figure one gear wheel is turning another in the usual way. If T is the number of teeth on the larger wheel and N is the number of turns this wheel is making per minute, while t is the number of teeth on the smaller wheel and n is the number of turns this wheel is making per minute, then we have the following ***law of gear wheels:***

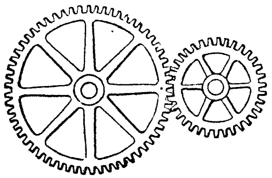


FIG. 53.

$$\frac{T}{t}=\frac{n}{N}, \text{ or } TN=tn.$$

EXERCISES — GEAR WHEELS

1. If the large wheel in Fig. 53 has 60 teeth and is making 100 revolutions per minute, how many revolutions per minute is the smaller wheel making if it has 20 teeth?

[HINT. By the law stated above, we have $60/20 = n/100$.]

2. A wheel of 90 teeth that is making one revolution per second, fits into a smaller wheel which is revolving twice as fast. How many teeth are there in the smaller wheel?

3. In order that the small wheel may always revolve just twice as fast as the large one, how should the wheels be made? Why?

[HINT. The example supposes simply that $n=2N$.]

4. A certain gear wheel of T teeth is making N revolutions per second. It fits into another similar wheel having t teeth. Show that if the latter wheel be replaced by one having r more teeth, the new one will revolve slower than the old one by the amount

$$\frac{TNr}{t(t+r)} \text{ rev. per sec.}$$

114. Belts. In Fig. 54, one wheel is turning another by means of a belt in the usual manner. If D is the diameter of the larger wheel and N is the number of turns this wheel is making per minute, while d is the diameter of the smaller wheel and n is the number of turns this wheel is making per minute, then we have the following ***law of belts:***

$$\frac{D}{d}=\frac{n}{N}, \text{ or } DN=dn.$$

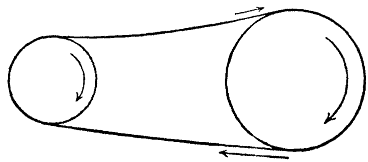


FIG. 54.

EXERCISES — BELTS

1. Show that if the two wheels are of the same size, each makes the same number of turns per minute.

[HINT. This supposes $D=d$.]

2. Show that if the diameter of the small wheel is one third that of the large one, it will make three times as many turns per minute.

3. A wheel whose diameter is 10 inches is revolving at the rate of 1 turn per second and is belted to a smaller wheel whose diameter is 3 inches. By how many turns per minute will the smaller wheel be slowed down if its diameter be increased by 2 inches?

4. A wheel of diameter D is making N revolutions per minute, and is belted to another wheel whose diameter is d. Show that in order to increase the speed of the latter wheel by q revolutions per minute, it would be necessary to diminish its diameter by

$$\frac{DNq}{n(n+q)}.$$

115. Similar Figures. When two geometric figures have exactly the same shape (though not necessarily the same size) they are called ***similar figures.*** Thus, any two circles are similar figures, as likewise any two squares, or spheres, or cubes.

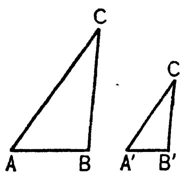


FIG. 55.

Two triangles may be similar, as illustrated in the figure. Note that, though not of the same size, these triangles have exactly the same *shape.*

The following facts are shown in Geometry to be true of any two similar figures.

(*a*) *Corresponding lines are proportional.*

Thus, in the two similar triangles above, if the side AB of the one triangle is twice as long as the corresponding side $A'B'$ (read A *prime*, B *prime*) of the other triangle, then BC is twice as long as $B'C'$; that is,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

In the same way, we have also

$$\frac{AB}{A'B'} = \frac{CA}{C'A'}.$$

(*b*) *Areas are proportional to the* ***squares*** *of corresponding lines.*

Thus, if one circle has a radius of length R and another circle has a radius of length r, the area A of the first circle is to the area a of the second as R^2 is to r^2; that is, we have the proportion

$$\frac{A}{a} = \frac{R^2}{r^2}.$$

Compare Ex. 15, p. 182.

(*c*) *Volumes are proportional to the* ***cubes*** *of corresponding lines.*

Thus, if one sphere has a radius of length R and another sphere has a radius of length r, the volumes V and v of the spheres are such that we have the proportion $V/v = R^3/r^3$.

EXERCISES — SIMILAR FIGURES

1. In the two similar triangles shown in § 115 suppose $AB = 1$ foot, $A'B' = 8$ inches, and $BC = 1\frac{1}{2}$ feet. How long must $B'C'$ be?

[HINT. Measure all lengths in inches and let x be the length of $B'C'$. Then by (*a*), § 115, we have $12/8 = 18/x$.]

2. If a tree casts a shadow 50 feet long when a post 4 feet high casts a shadow 5 feet long, how high must the tree be?

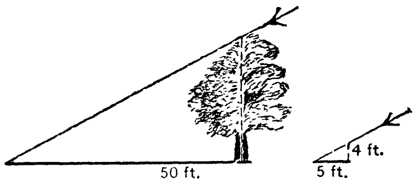


FIG. 56.

3. A triangle has its sides 3 inches, 4 inches, and 5 inches long. Another triangle of the same shape has its shortest side 2 inches long. What are the lengths of the other sides of this triangle?

4. Of the two triangles in Ex. 3, the first has an area of 6 square inches. What is the area of the second?

[HINT. Let x be the area of the second. Then, by (*b*) of § 115, we have

$$\frac{6}{x} = \frac{3^2}{2^2},$$

that is,

$$\frac{6}{x} = \frac{9}{4}.$$

5. Compare the areas of two city lots of the same shape if a side of the one is twice as long as the corresponding side of the other. Does your answer apply no matter what the shape is so long as it is the *same* for each lot?

6. What is the effect upon the area of a circle of trebling its diameter?

[HINT. Let D be the first diameter. Then $3D$ will be the second diameter.]

7. Compare the volume of a sphere whose diameter is 1 inch with that of a sphere whose diameter is 2 inches.

8. If a bottle of a certain shape holds 1 pint, how much will a similar bottle half as high hold?

9. A man whose eye is 5 ft. 6 in. above the ground sights over the top of a 12-foot pole and just sees the top of a tree. If he is 7 ft. from the pole and 63 ft. from the tree, how high is the tree?

[HINT. First draw a figure.]

10. The figure represents a kind of compasses used by draftsmen. By adjusting the screw at O, the lengths OA and OC, and the corresponding lengths OB and OD, may be changed proportionally. If $OA = 3$ in. and $OC = 5$ in., what part of the opening CD will the opening AB be?

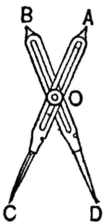


FIG. 57.

116. Mean Proportional. If the means of a proportion are equal, either mean is called the ***mean proportional*** between the extremes.

Thus, in the proportion $1:2=2:4$, we have 2 as the mean proportional between 1 and 4. Likewise, in the proportion

$$18:6=6:2,$$

6 is the mean proportional between 18 and 2, and in the proportion

$$x^4 : x^3 = x^3 : x^2,$$

we have x^3 as the mean proportional between x^4 and x^2.

EXERCISES — MEAN PROPORTIONAL

1. Find the mean proportional between 6 and 24.

SOLUTION. Let x be the mean proportional. Then, $6/x = x/24$. Whence, $x^2 = 144$, and $x = 12$. *Ans.*

Find the mean proportional between the two numbers given in each of the following exercises.

2. 3 and 27.	**4.** $4\,a$ and $9\,a$.	**6.** 1 and $(a+b)^2$.
3. 50 and 1.	**5.** $2\,r^3$ and $2\,r$.	**7.** $4\,mn$ and $9\,m^3n$.

8. In the semicircle ABC, suppose a line CD drawn perpendicular to AB. Then (as shown in geometry) the length of CD will be a mean proportional between the lengths AD and DB.

If $AD = 2$ inches and $DB = 18$ inches, find CD.

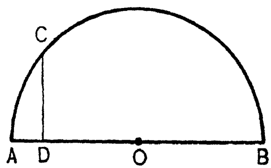


FIG. 58.

9. In Fig. 58, suppose $AB = 29$ feet, and $AD = 4$ feet. What is CD?

10. The figure shows a circle and a point P outside it from which are drawn two lines PT and PS. The first of these lines just *touches* the circle and is called a ***tangent***, while the second line cuts through the circle at two points R and S and is called a ***secant***. In all such cases, the tangent PT is a mean proportional between the whole secant PS and its external part PR (as shown in geometry).

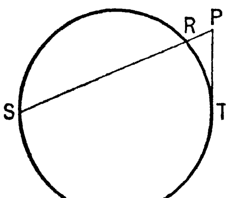


FIG. 59.

Find the length of PT if $PR = 9\frac{3}{5}$, and $RS = 50\frac{2}{5}$.

CHAPTER XII

GRAPHICAL REPRESENTATION

117. Use of Diagrams. A few examples will show how diagrams are often used in everyday life to bring facts clearly before the eye.

EXAMPLE 1. A branch of the Y. M. C. A. wished to let people know of its progress in collecting money for a new

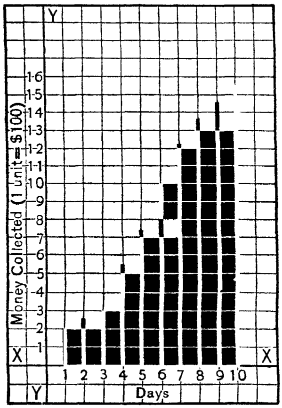


FIG. 60.

building. It placed a large signboard on the street and after ten days the board had the appearance shown in Fig. 60.

EXPLANATION. Two lines XX and YY had been drawn perpendicular to each other and each had been divided into equal units, beginning at the point where the lines cross. The points of division

were numbered 1, 2, 3, etc., as on a yardstick. Each unit on XX represented one of the days during which the money had been coming in, while each unit on YY represented \$100. Starting at the point marked 1 on XX, the secretary of the Y. M. C. A. had drawn a heavy line extending upward until its end was on the level with the point marked 2 on YY. This indicated that on the *first* day just \$200 was received. Similarly, he had drawn a heavy line beginning at the point marked 2 on XX and extending upward $2\frac{1}{2}$ units as measured on YY. This indicated that on the *second* day the amount received was \$250. In the same way, he had drawn a heavy line upward corresponding to each of the 10 days. Read for yourself (from the scale on YY) the amount received on each day after the second.

EXAMPLE 2. Figure 61 shows the home expenses of a small family for a period of twelve months, beginning with January.

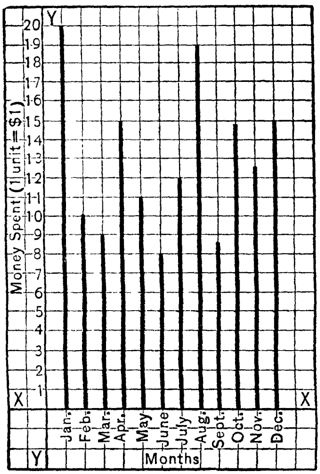


FIG. 61.

Each unit on XX represents 1 month, while each unit on YY represents \$1. Read for yourself the amount spent during each of the twelve months.

EXAMPLE 3. Figure 62 shows the number of miles of railway built in the United States from 1881 to 1890. Each unit on XX represents a year (beginning with 1881), while each unit on YY represents 500 miles. Read off (as near as you can) the number of miles built during each of the nine years

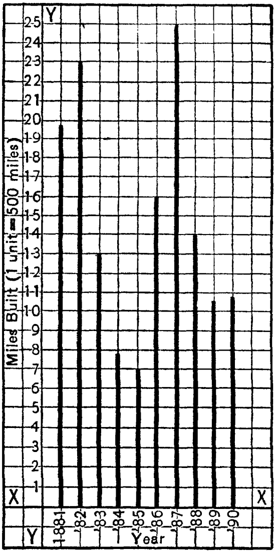


FIG. 62.

EXERCISES†

In the three Examples of § 117 a diagram was given each time and you were asked to read off from it the facts which it expressed. In the Examples below this is reversed; that is, the facts are given first in a table and you are asked to draw the corresponding diagram yourself.

1. The table below shows the money spent each month of the year by a small family for clothing. Draw a diagram (similar to Fig. 61) to represent the facts here given.

Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
$20	$10	$12	$20	$15	$8	$9	$20	$7	$20	$28	$10

2. The miles of railway built in the United States from 1891 to 1900 is shown in the table below. Draw a diagram (similar to Fig. 62) to represent this table, indicating clearly the numerical values in the margin, as in Fig. 62.

† The pupil will find it to his advantage to secure at this point paper ruled in small squares. Such paper is called *squared paper*, or *coördinate paper*. It may be found at most stationery stores.

Year	1891	1892	1893	1894	1895	1896	1897	1898	1899	1900
Miles	4000	4500	2500	2000	1500	1600	1800	2200	3800	3500

3. Draw a diagram to represent the temperature record given in the table below. Each unit on XX should represent 1 hour, beginning with 7 A.M., while each unit on YY should represent 1°, beginning with 22°.

Time	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 M.	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.
Temperature	22°	24°	26°	29°	34°	38°	42°	41°	39°	35°	30°	25°

4. Draw a diagram similar to that for Ex. 3 to represent the temperature record below. Observe that some of the temperatures here are *negative*. Wherever this happens, draw the corresponding heavy line *downward*, instead of upward.

Time	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 M.	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.
Temperature	+2°	+3°	+5°	+7°	+9°	+8°	+6°	+3°	0°	−5°	−8°	−12°

5. The table for *dry measure* as learned in arithmetic is

2 pints = 1 quart, 8 quarts = 1 peck, 4 pecks = 1 bushel.

Draw a diagram for this, placing the words *pint*, *quart*, *peck*, *bushel*, each one unit apart, along XX, and letting each unit on YY represent 1 quart.

6. Draw a diagram for the table learned in arithmetic for *liquid measure*. Proceed as in Ex. 5, taking each unit on YY to represent 1 quart. It is interesting to compare the two diagrams.

118. The Graph. Thus far we have considered diagrams consisting of a series of heavy upright lines. In practice, we do not always draw in the full length of such lines, but we simply mark (by a cross or a dot) their *end* points, and then draw a smooth line or curve through all such points. This gives what is known as a **graph**. It will be illustrated by the following examples.

EXAMPLE. The average weight of boys at different ages beginning at 6 years and continuing up to 15 years is given in the table below.

Age	6	7	8	9	10	11	12	13	14	15
Weight	50	53	57	62	67	72	78	85	93	105

Draw the graph of this table.

SOLUTION. We begin just as in preparing the diagrams in § 117, letting each unit on XX here represent a year, beginning with 6, and letting each unit on YY represent 10 pounds, beginning with 50. But instead of drawing in the heavy upright lines, we simply mark (by the cross) their upper end points, and then draw, free hand, a smooth curve through all these points, giving the result shown in Fig. 63. This curve is the *graph* of the above table.

119. Use of the Graph. The graph in Fig. 63 not only gives us a picture of the weight of boys at exactly 6 years, 7 years, etc., but also at intermediate ages such as 7.5 years, 8.75 years, etc. For example, at 7.5 years the weight is seen to be about 55 pounds, because when we are at the point 7.5 on XX the height of the curve (as measured on YY) is seen to be about 55. Likewise, for any age whatever between 6 and 15 years the graph gives us instantly a good idea of the corresponding average weight. By the use of such a curve it is easy to tell whether a boy is under or over average weight.

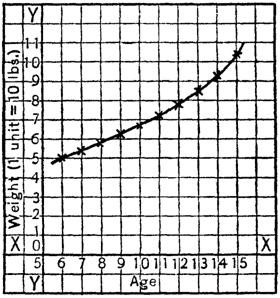


Fig. 63. — Average Weight of Boys.

Note also that the graph grows *steeper* as it goes to the right, thus indicating that the *rate of growth* steadily increases as the boy gets to be near 15 years of age.

EXERCISES — GRAPHS

1. The table below shows the average weight of girls between the ages of 6 and 15 years. Draw the graph for this table similar to that for boys shown in § 118.

Age	6	7	8	9	10	11	12	13	14	15
Weight	43	47	52	57	62	69	78	89	98	106

2. On the figure which you have drawn for Ex. 1 sketch in also the curve for boys (as given in Fig. 63). In other words, draw both graphs on the *same* sheet, using the same lines XX and YY for both. Upon comparing the two curves you will see that they cross each other at a certain point, owing to the fact that the curve which starts lowest rises fastest. What is thus brought out about the comparative growths of boys and girls?

3. Draw the graph showing how one's savings increase if he saves \$1.00 a month for 12 months.

[HINT. Each unit on XX represents a month and each unit on YY represents \$1.00. The graph turns out to be a straight line.]

4. Draw a graph showing how one's savings increase if he saves \$1 the first month, \$2 the second month, \$3 the third month, and so on for a year. Compare your result with that for Ex. 3 and state what facts are thus brought out to the eye.

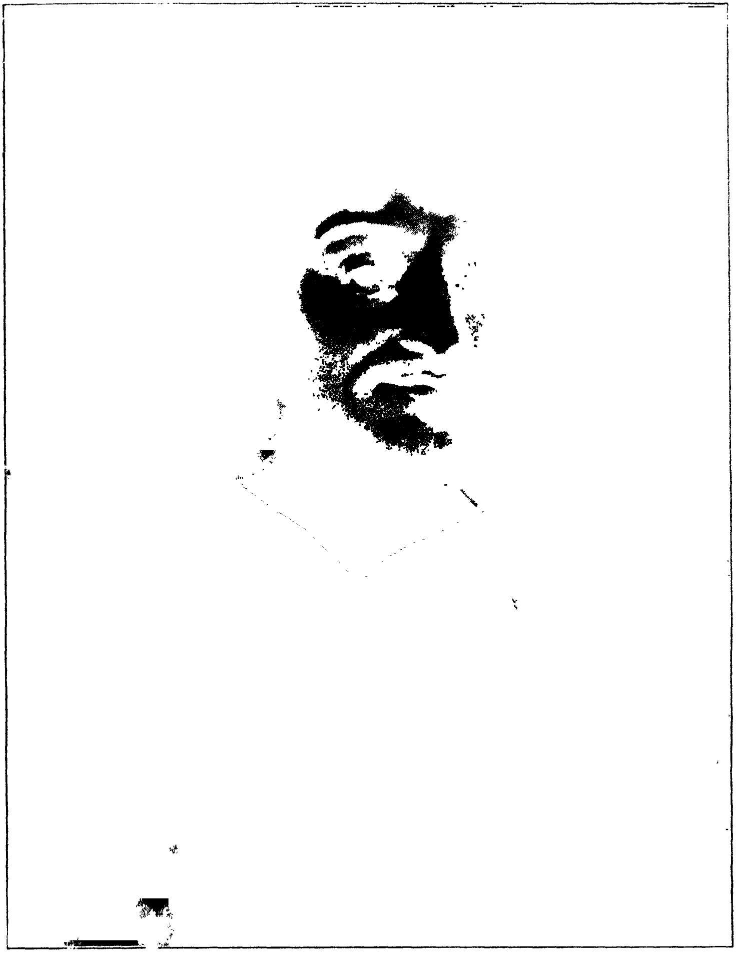
5. The water in a glass tube was at 60°. Heat was applied and the temperature of the water was then measured at intervals of five minutes, giving the results shown in the table below.

Minutes	0	5	10	15	20	25	30	35	40
Degrees	60	68	76	83.2	89.6	95.5	101	106	110

Draw the corresponding graph and from it answer (approximately) the following questions: (*a*) What was the temperature at the end of 12 minutes? (*b*) At the end of 27 minutes?

6. Below are given the times of sunrise at intervals of 15 days for a certain period of the year. Draw a curve (similar to that in § 118) to show its change during that period. Represent the dates along the line XX with 1 unit for every 15 days. Take each unit on the line YY equal to 25 minutes.

Date	Nov. 17	Dec. 1	Dec. 16	Dec. 31	Jan. 14	Jan. 29	Feb. 13
Time of Sunrise	6:54	7:09	7:24	7:30	7:27	7:16	6:59



DESCARTES

(*René Descartes*, 1596–1650)

Profound student and ranked as one of the greatest leaders of all time in both mathematics and philosophy. He invented representation by graphs and was thus led to the discovery and development of the branch of mathematics called Analytic Geometry. He was also much interested in medicine and surgery.

120. The Graph of an Equation. If \$100 be invested at 3% simple interest, the interest amounts at the end of 1 year to \$3, at the end of 2 years to \$6, at the end of 3 years to \$9, etc. This leads to the table below.

Year	1	2	3	4	5	6	7
Interest	\$3	\$6	\$9	\$12	\$15	\$18	\$21

Figure 64 shows the graph drawn from this table as in § 118, with the years indicated on the horizontal line XX and the total interest on the vertical line YY.

There is another way of considering this matter. Since the interest amounts at the end of 1 year to \$3, at the end of 2 years to \$6, at the end of 3 years to \$9, and so on, we see that if i stands for the interest at the end of t years, we have always the equation

$$i = 3\,t.$$

This equation gives in condensed form all that the above table and graph really say, for if we put $t = 1$ in the equation we get (as we should) $i = 3$, likewise, if we put $t = 2$ in the equation it gives $i = 3 \cdot 2$, or 6, which agrees with the graph, etc. Thus, we have here an equation and a graph which *correspond* to each other. Whenever this happens the graph is said to be ***the graph of the equation.***

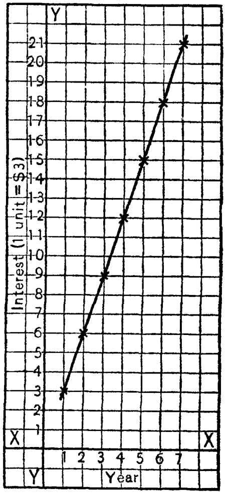


FIG. 64.

In such a case, the graph and the equation represent the same facts, but in two entirely different ways.

EXERCISES — GRAPHS OF EQUATIONS

1. Draw the graph like that of § 120, showing the interest on \$100 at 2%, and write the corresponding equation.

2. Draw the graph like that of § 120, showing the interest on \$100 at 4%, and write the corresponding equation.

3. Draw the graph of the equation $i = 5\,t$.

[HINT. First let t take the values 1, 2, 3, etc., and form a table. Then proceed as in § 120.]

4. A man leaves a certain place and walks at the rate of $2\frac{1}{2}$ miles per hour. Construct a graph to show how far he has walked after any given length of time, and find the corresponding equation.

[HINT. Let s stand for the distance walked in t hours, and see § 107.]

5. If a small heavy body, as a bullet, is dropped vertically downward from some high place, as a tower, the distance s (in feet) which it has passed over by the end of t seconds is determined by the equation $s = 16\,t^2$. Draw the graph of this equation, letting a unit on the line XX represent 1 second, and a unit on the line YY represent 16 feet.

[HINT. The graph is *not* a straight line.]

121. Definitions. Two lines perpendicular to each other (such as the lines XX and YY of § 117) are called a pair of ***axes***. The horizontal line XX is called the ***axis of x***, while the vertical line YY is called the ***axis of y***. The point where the two lines meet (or cross) is called the ***origin***.

Just as in Figure 18, page 34, we used the + sign with numbers to the *right* of the origin (or zero point) and the − sign with numbers to the *left* of that point, so we shall hereafter regard all distances on the x-axis (XX in Fig. 65)

as + if measured to the *right* of the origin, and as − if measured to the *left* of that point. Moreover, we shall carry out this same idea along the y-axis (YY in Fig. 65), thus considering as + all distances *upward* from the origin, and as − all distances *downward* from that point.

Suppose we now think of a point which is not on either axis, as the point P in Fig. 65. The distance from P to the y-axis is seen to be 4 units (as measured along the x-axis below), and the distance from P to the x-axis is seen to be 3 units (as measured on the y-axis). More exactly, these measurements are +4 and +3 respectively, since the first is to the *right*, while the second is *upward*. In this way we may describe the location of P very briefly by calling it the point (+4, +3). In the same way, the point marked Q in the figure is the point (−5, 4) because it is located 5 units to the *left* of the y-axis and 4 units *upward*. Again, R as it appears in the figure is the point (−4, −4) (Why?), while S is the point (6, −2) (Why?).

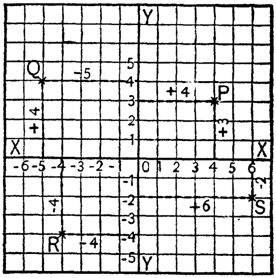


FIG. 65.

Observe carefully that the symbol for a point thus becomes in all cases (x, y) where x and y are numbers (positive or negative) the *first* of which gives the distance of the point from the y-axis, while the *second* gives the distance from the x-axis. The numbers x and y are called the ***coördinates*** of the point; the first is sometimes called the "*x of the point*" and the second is called the "*y of the point.*"

NOTE. If the point is *on* the x-axis its distance from that axis is 0, hence such a point is represented by $(x, 0)$. Similarly, a point *on* the y-axis becomes $(0, y)$.

EXERCISES — COÖRDINATES

1. Locate on squared paper (plot) the points $(-3, -1)$; $(2, -4)$; $(-1, 2)$; $(3, 4)$; $(-3, 7)$.

2. Plot the points $(2\frac{1}{2}, -3)$; $(4, -5)$; $(-3, -6)$; $(-4, 2)$.

3. Plot the points $(2, 1)$, $(-8, 1)$, and $(1, -6)$. Join these points by straight lines. What sort of figure is formed? Find the coördinates of the middle point of the horizontal side.

4. A certain street runs due east and west. It is met by another street which runs due north and south, thus forming a "four corners." Taking the meeting point as origin and the east and north directions as the $+XX$ and $+YY$ axes, respectively, what are the coördinates of a flagpole which stands due southeast from the origin at a distance of 100 feet from each road? Answer the same question when the pole stands 100 feet due south of the crossing point.

5. Plot each of the following points and then see if it is possible to draw a straight line passing through all of them: $(1, 2)$; $(2, 4)$; $(0, 0)$; $(-1, -2)$; $(-2, -4)$; $(-3, -6)$; $(-\frac{1}{2}, -1)$.

122. The Complete Graph of an Equation. The graph of an equation, as explained in § 120, becomes much more complete when we use a pair of axes such as we have in § 121. This will appear from the following example.

Example. Plot the graph of the equation $2x+y=6$.

SOLUTION. First draw a pair of axes. Now give x any value in the equation; for example, let $x=1$. The equation then becomes $2\times 1+y=6$, or simply $2+y=6$, from which, upon solving for y, we get $y=4$. Thus, the *pair* of values $(x=1, y=4)$ when used together satisfy the given equation.

We now plot the point $(1, 4)$ since it is what corresponds

(as regards the axes) to the pair of values just found. The point (1, 4) is thus said to "belong to" the given equation. See P_1 in the figure.

Next, give x some other value; for example, $x = 2$. When this is done in the equation we find that $y = 2$. Explain this. Therefore, the point (2, 2) is another point which belongs to the equation, and is now to be plotted. See P_2 in the figure.

We proceed to plot several points in this way, first giving x a value in each case and then getting the corresponding y from the equation. In doing this it is to be observed that x may be given negative as well as positive values and y also may become negative, but we can plot the point in every case, as shown in § 121. The table below shows a variety of values of x and the corresponding value of y for each value of x. Note the signs. For example, when $x=4$, $y=-2$; when $x=-1$, $y=8$; etc.

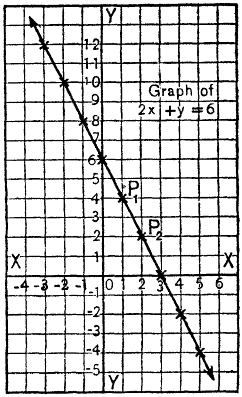


FIG. 66.

x	1	2	3	4	0	-1	-2	-3	-4
y	4	2	0	-2	6	8	10	12	14

The graph of the given equation is now obtained by drawing the straight line which passes through *all* the points thus plotted (see Fig. 66). This line may be extended indefinitely in either direction (see the arrows in the figure) and when so extended it constitutes the *complete* graph of the equation $2x+y=6$.

123. Linear Equations. Any equation which contains two letters (usually x and y) is called ***linear*** if these letters occur to no higher power than the first. Thus, the equation $2x+y=6$ considered in § 122 is a linear equation in x and y; so also are the equations $6x-3y=2$, $\frac{1}{2}x=3y-1$, etc.; but $x^2-y^2=1$ is *not* linear.

The graph of every linear equation is a straight line. It can be shown that if we take *any* point on this line its coördinates will satisfy the given equation.

EXERCISES—GRAPHS OF LINEAR EQUATIONS

Draw the graph of each of the following linear equations. In each case a table should be formed as in the Example of § 122. Plot at least three points in each case.

1. $x+y=4$.	**3.** $x+2y=6$.	**5.** $x+3y=3$.
2. $x-y=3$.	**4.** $2x+3y=1$.	**6.** $4x+3y=12$.

124. Plotting an Equation by Means of Two Points. We need find only two points to plot the line through them. Any two points will do, but it is usually most convenient to use the two points where the line crosses the axes.

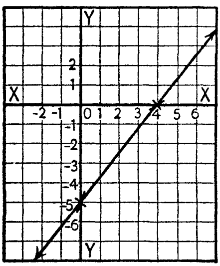


FIG. 67.

EXAMPLE. Plot the equation

$$5x-4y=20.$$

SOLUTION. Placing $x=0$ in the equation, gives $y=-5$.
Placing $y=0$ in the equation, gives $x=4$.

Plotting the two points thus obtained, namely $(0, -5)$ and $(4, 0)$, (which, by the Note in § 121 are the points where the graph crosses the axes) and drawing (with a ruler) the line through them, we get the graph required, as shown in Fig. 67.

EXERCISES — PLOTTING STRAIGHT LINES BY TWO POINTS

Draw the graph of each of the following linear equations by plotting two points on it.

1. $3x-2y=-6$. **3.** $3x-2y-6=0$. **5.** $2x-5y=10$.

2. $3x+2y=-6$. **4.** $3x=2y$. **6.** $4x=7y-14$.

125. Simultaneous Equations. When we draw the graphs of the equations

$$x+y=6,$$

and

$$3x-2y=-2,$$

we find that the lines intersect (cut each other) in one point, namely (2, 4). Since this point is thus on *both* lines, it follows that the corresponding values, $x=2$, and $y=4$, must satisfy *both* equations. Test this and see that it is true. A set of values, such as (2, 4), that satisfies both of the two equations is said to be a ***solution*** of the equations.

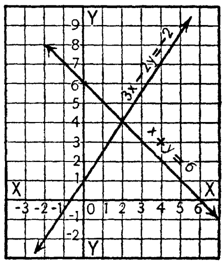


FIG. 68.

Any two equations considered together in this way are called ***simultaneous equations.***

126. Inconsistent Equations. If we draw the graphs of the simultaneous equations

$$\begin{cases} x+y=3, \\ x+y=6, \end{cases}$$

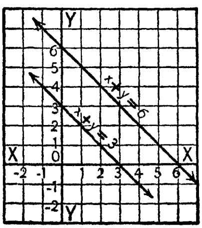


FIG. 69.

we find that the lines do not intersect; in other words, the lines are *parallel.* Thus, the equations have no pair of values in common. Such a pair of simultaneous equations is called ***inconsistent.***

EXERCISES — SIMULTANEOUS EQUATIONS

Draw the graphs of each of the following pairs of simultaneous equations. Determine which have a solution and which are inconsistent. Whenever there is a solution, find (from the graph) what it is and test your result in the equations.

1. $\begin{cases} x+2y=3, \\ 2x+y=3. \end{cases}$

2. $\begin{cases} x-2y=6, \\ 2x-4y=8. \end{cases}$

3. $\begin{cases} 3x+2y=6, \\ 3x-2y=6. \end{cases}$

4. $\begin{cases} 3x+2y=6, \\ 3x+2y=-6. \end{cases}$

5. $\begin{cases} 3x+2y=6, \\ 3x-2y=-6. \end{cases}$

6. $\begin{cases} y=-3, \\ 3x-2y=-6. \end{cases}$

7. $\begin{cases} 2x-y=4, \\ 4x-2y=12. \end{cases}$

8. $\begin{cases} 2x+3y=13, \\ x=-2. \end{cases}$

9. $\begin{cases} 2x-4y=-5, \\ 4x+2y=5. \end{cases}$

10. $\begin{cases} 4x-y=0, \\ 3x+y=7. \end{cases}$

11. $\begin{cases} x-y=0, \\ x+y=0. \end{cases}$

12. $\begin{cases} x+2y-3=0, \\ 2x-y+1=0. \end{cases}$

***127. Graphical Study of Motion.** The way in which motion may be studied graphically is best seen from one or two examples.

EXAMPLE 1. An automobile starts out and travels at the rate of 20 miles an hour, and two hours later another starts from the same place and goes in the same direction at 30 miles an hour. How long before the second overtakes the first?

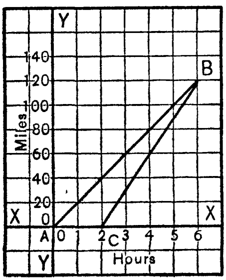


FIG. 70.

SOLUTION. Take each unit on XX to represent 1 hour and each unit on YY to represent 20 miles. The motion of the first automobile is then represented by a straight line starting at the origin and rising 1 unit for every unit it goes to the right; that is, it is represented by the line AB.

As for the second automobile, it does not start until 2 hours have elapsed and then goes at the rate of 30 miles an hour. This means that its motion is represented by a straight line starting

at the 2-hour mark on XX and then rising $1\frac{1}{2}$ units for every unit it goes to the right; that is, it is represented by the line CB.

The point B where these two lines meet corresponds graphically to the meeting of the automobiles, because at this point both have traveled the same distance as measured off on YY. The time that goes with this point, as measured off on XX, is seen to be 6 hours. Therefore, the second automobile will overtake the first one 6 hours after the first one starts, or, 4 hours after the second one starts. *Ans.*

Example 2. Two automobiles start from two towns that are 110 miles apart and travel towards each other, the one at 20 miles an hour and the other at 30 miles an hour. How long before they will be 10 miles apart?

Solution. Take the same units as in Example 1. Then the two motions will be represented by the two lines in Fig. 71. Note that the line representing the machine traveling at 20 miles per hour starts at 0 hours and rises 1 unit for every unit it goes to the right; while the other line, which represents the machine traveling at 30 miles per hour, starts at the 0 hour also, but at the 110 mark, and it descends $1\frac{1}{2}$ units for every unit it goes to the right.

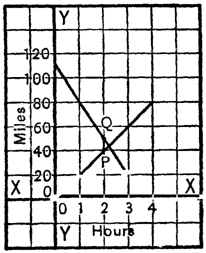


Fig. 71.

Now, the question is, "When will the two machines be 10 miles apart?" Graphically, this means "What is the point on XX where the difference between the distance measured up to the lines will be 10 miles; that is, $\frac{1}{2}$ a unit?" This difference is represented by PQ. The point on XX is seen to be 2. Hence, the answer to the example is 2 hours.

* EXERCISES — MOTION

1. A boy starts from home at noon and rides on a bicycle 10 miles an hour toward a certain town. At 3 o'clock in the afternoon a man starts out from the same place in an automobile and travels the same road at 30 miles an hour. Determine graphically how soon the boy will be overtaken, and also how far he will then be from home.

2. Answer the following question graphically: In a bicycle race

A goes 22 yards per second, and B goes 12 yards per second. B has a start of 50 yards. Who will win the race?

[HINT. Let each unit on XX represent 1 second, and each unit on YY represent 10 yards.]

3. Solve the following graphically: A man can row 18 miles per hour downstream, and 2 miles per hour returning. How far downstream can he go if he wishes to be back home in 10 hours from the time he starts out?

CHAPTER XIII

SIMULTANEOUS EQUATIONS SOLVED BY ELIMINATION

128. Solution of Simultaneous Equations by Elimination. The solution of two simultaneous equations (such as those considered in § 125) may be readily found by a process called *elimination*. As there are several kinds of elimination, we shall first consider that which is called ***elimination by substitution***.

EXAMPLE 1. Solve the simultaneous equations

(1) $$x+y=6,$$

(2) $$3x-2y=-2.$$

SOLUTION. From equation (1) we have (by transposing the x)

$$y=6-x.$$

If we substitute (or place) this value of y in equation (2), we obtain the equation

$$3x-2(6-x)=-2,$$

which is an equation containing only the *one* letter, x; in other words, we have obtained from (1) and (2) an equation in which the y has been *removed* or ***eliminated***.

This last equation (being like those considered in the earlier Chapters) may be at once solved. Thus, it reduces to

$$3x-12+2x=-2, \text{ or } 5x=10,$$

which gives $$x=2.$$

To get y, substitute the value $x=2$ in (1). The result is

$$2+y=6, \text{ or } y=4.$$

The required solution of (1) and (2) is therefore $(x=2, y=4)$. *Ans.*

CHECK. When $x=2$ and $y=4$, the first member of equation (1) becomes $2+4$, which reduces (as this equation demands) to 6. Similarly, when $x=2$ and $y=4$ the first member of (2) becomes $3\times2-2\times4=6-8=-2$, as required.

EXAMPLE 2. Solve the simultaneous equations

(1) $$2x+3y=4,$$
(2) $$5x-2y=-9.$$

SOLUTION. From (1) we have

$$3y=4-2x,$$

and therefore

$$y=\frac{4-2x}{3}.$$

Substituting this value of y in (2) gives

$$5x-2\left(\frac{4-2x}{3}\right)=-9.$$

This is an equation for the *one* letter x, and may therefore be solved. Thus, clearing it of fractions we obtain

$$15x-2(4-2x)=-27,$$

or,

$$15x-8+4x=-27,$$

which gives

$$19x=-19,$$

and hence

$$x=-1.$$

Placing $x=-1$ in equation (1) gives

$$-2+3y=4,$$

or,

$$3y=6.$$

Hence,

$$y=2.$$

The required solution of (1) and (2) is therefore $(x=-1,\ y=2)$. *Ans.*

CHECK. When $x=-1$ and $y=2$, the first member of (1) becomes $2\times(-1)+3\times2$, which reduces (as this equation demands) to 4. Similarly, when $x=-1$ and $y=2$, the first member of (2) becomes $5\times(-1)-2\times2$, which reduces to -9, as required.

NOTE. Observe that the way which we have just used for solving two simultaneous equations gives their solution without requiring us to draw the graph. For this reason it is usually shorter than the method studied in § 125, though the latter is very valuable in making clear what the solution means.

From the two examples worked above we obtain the following rule.

RULE FOR SOLVING TWO SIMULTANEOUS EQUATIONS BY SUBSTITUTION. *Solve the first equation so as to get y in terms of x. Place the result in the second equation, which may then be solved to get the desired value of x. After x has been found, put its value in the first equation, which may then be solved to get the desired value of y.*

NOTE. This rule may be stated in a more general form as follows: Solve either equation for one of the letters in terms of the other one. Place the result in the other equation and solve. Having thus found the value of one of the letters, the value of the other may be found by substituting the known value into either of the given equations and solving the resulting equation.

EXERCISES

1. Solve Example 1 of § 128 by the method of § 125 and compare the solution you thus obtain with that already worked out in § 128. They should agree.

Solve each of the following pairs of simultaneous equations by substitution and check your answers.

2. $\begin{cases} x+y=7, \\ x-y=1. \end{cases}$

3. $\begin{cases} x+3y=11, \\ 3x+y=9. \end{cases}$

4. $\begin{cases} x+y=12, \\ 3x+y=24. \end{cases}$

5. $\begin{cases} x+y=30, \\ 3x-2y=25. \end{cases}$

6. $\begin{cases} 3r-7s=40, \\ 4r-3s=9. \end{cases}$

7. $\begin{cases} 3p+2q=26, \\ 5p-2q=38. \end{cases}$

8. $\begin{cases} 5u+7v=125, \\ 7u-v=13. \end{cases}$

9. $\begin{cases} 2m-3n=-14, \\ 3m+7n=48. \end{cases}$

10. $\begin{cases} 9X-4Y=3, \\ 7X+2Y=33. \end{cases}$

11. $\begin{cases} 2s-3t=12, \\ 3s+5t=8. \end{cases}$

12. $\begin{cases} 2x-3y=0, \\ 3x+y=11. \end{cases}$

13. $\begin{cases} 4x+3y=1, \\ 3x-2y=\frac{13}{6}. \end{cases}$

For further exercises on this topic, see Appendix, p. 308.

129. Elimination by Addition or Subtraction. Suppose that we have the simultaneous equations

$$x+2y=4,$$
$$3x-2y=2.$$

Here the coefficients of y in the two equations are numerically equal, but opposite in sign, so that if we add the equations together we obtain an equation containing only x. Thus,

$$\begin{array}{rl} & x+2y=4 \\ & \underline{3x-2y=2} \\ \text{Adding,} & 4x \qquad =6. \end{array}$$

This example illustrates what is called ***elimination by addition.***

Again, suppose that we have the simultaneous equations

$$4x+2y=10,$$
$$x+2y=4.$$

Here the coefficients of y are the same numerically and in sign so that if we subtract the second equation from the first we obtain an equation containing only x. Thus,

$$\begin{array}{rl} & 4x+2y=10 \\ & \underline{x+2y=4} \\ \text{Subtracting,} & 3x \qquad =6. \end{array}$$

This illustrates ***elimination by subtraction.***

The way in which elimination by addition or subtraction is used to solve any two simultaneous equations will be seen from the following examples:

Example 1. Solve the simultaneous equations

(1) $$3x-2y=-1,$$
(2) $$2x+y=11.$$

Solution. First multiply both members of (2) by 2, obtaining,

(3) $$4x+2y=22.$$

Adding (1) and (3)

$$\begin{array}{r} 3x-2y=-1 \\ \underline{4x+2y=22} \\ 7x \qquad =21; \end{array}$$

hence

$$x=3.$$

Substituting $x=3$ in (1) gives

$$9-2y=-1,$$

or,

$$-2y=-10;$$

whence,

$$y=5.$$

It follows that the desired solution is $(x=3,\ y=5)$. *Ans.*

CHECK. When $x=3$ and $y=5$ the first member of (1) becomes $9-10$ which, as the equation demands, is equal to -1. Similarly, the first member of (2) becomes $6+5$, which equals 11, as required.

EXAMPLE 2. Solve the equations

(1) $$3x-2y=8,$$

(2) $$4x-3y=10.$$

SOLUTION. Multiplying (1) by 3 gives

(3) $$9x-6y=24.$$

Multiplying (2) by 2 gives

(4) $$\underline{8x-6y=20.}$$

Subtracting gives

$$x=4.$$

Substituting $x=4$ in (1) gives

$$12-2y=8,$$

or,

$$-2y=-4,$$

or,

$$y=2.$$

The desired solution is therefore $(x=4,\ y=2)$. *Ans.*

CHECK. When $x=4$ and $y=2$, the first member of (1) becomes $12-4$, which is 8 as required. Similarly, the first member of (2) becomes $16-6$, which is 10 as required.

A careful study of the examples above leads to the following rule.

RULE FOR SOLVING TWO SIMULTANEOUS EQUATIONS BY ADDITION OR SUBTRACTION. *First multiply one or both of the equations by such numbers as will make the coefficients of one of the letters (say, y) numerically equal. Then eliminate by addition if the resulting coefficients have unlike signs, or by subtraction if they have like signs. Proceed from this point to get the value of one letter, after which the other letter may be obtained by substitution.*

EXERCISES

Solve each of the following pairs of simultaneous equations by the method of addition or subtraction, and check your answers. In the first five it is not even necessary to multiply through in order to get suitable coefficients; hence these may be done mentally.

1. $\begin{cases} x+y=3, \\ x-y=1. \end{cases}$

2. $\begin{cases} x+y=7, \\ x-y=1. \end{cases}$

3. $\begin{cases} x+2y=9, \\ x-y=0. \end{cases}$

4. $\begin{cases} x+y=8, \\ x-y=-2. \end{cases}$

5. $\begin{cases} x+\frac{1}{2}y=3, \\ 2x+\frac{1}{2}y=7. \end{cases}$

6. $\begin{cases} 3x+5y=4, \\ 9x+10y=10. \end{cases}$

7. $\begin{cases} 2r+3s=13, \\ r+2s=8. \end{cases}$

8. $\begin{cases} x+y=9, \\ x-y=1. \end{cases}$

9. $\begin{cases} 5K+3T=8, \\ 7K-3T=4. \end{cases}$

10. $\begin{cases} 8w+9v=99, \\ 5w-7v=24. \end{cases}$

11. $\begin{cases} 5m-4n=7, \\ 2m+3n=12. \end{cases}$

12. $\begin{cases} 5x-2y=11, \\ 3y+x=9. \end{cases}$

13. $\begin{cases} p+2\,q=7, \\ 3\,p-2\,q=7. \end{cases}$

14. $\begin{cases} 2\,x+3\,y=41, \\ 2\,y+3\,x=39. \end{cases}$

15. $\begin{cases} 3\,x-2\,y=0, \\ 5\,x+2\,y=8. \end{cases}$

16. $\begin{cases} 5\,r+2\,s=214, \\ r=21\,s. \end{cases}$

17. $\begin{cases} \frac{1}{2}\,x+\frac{1}{3}\,y=2, \\ \frac{3}{4}\,x+\frac{2}{3}\,y=2. \end{cases}$

18. $\begin{cases} .5\,x+.2\,y=66.1, \\ x-y=9. \end{cases}$

For further exercises on this topic, see Appendix, p. 308.

130. Simultaneous Equations Containing Fractions. If the equations contain fractions, it is usually best to clear of fractions and reduce to the forms found in §§ 128, 129.

Example. Solve the equations

(1) $$\frac{12}{x-2}-\frac{8}{y-5},$$

(2) $$\frac{10}{x-4}=\frac{15}{y-3}.$$

Solution. Simplifying (1) by clearing it of fractions (as in Chap. X) gives

$$12\,y-60=8\,x-16,$$

which reduces to

(3) $$8\,x-12\,y=-44.$$

Simplifying (2) gives

$$10\,y-30=15\,x-60,$$

which reduces to

(4) $$15\,x-10\,y=30.$$

Thus, the equations (1) and (2) are now in the forms

$$8\,x-12\,y=-44,$$
$$15\,x-10\,y=30.$$

Hence they can be solved by methods given in §§ 128, 129.

Solving we find $x=8$, and $y=9$. *Ans.*

Check. When $x=8$ and $y=9$, equation (1) becomes $\frac{12}{6}=\frac{8}{4}$, which is seen to be true. Likewise, equation (2) becomes $\frac{10}{4}=\frac{15}{6}$, which is also true.

EXERCISES

Reduce the following equations to the forms found in §§ 128, 129 and solve by either of the methods there explained. Check your answers for Exs. 1–6.

1. $\begin{cases} \dfrac{7r-15}{3}=s, \\ 2r-s=3. \end{cases}$

2. $\begin{cases} \dfrac{x}{8}=\dfrac{y}{9}, \\ 5x-3y=13. \end{cases}$

3. $\begin{cases} \dfrac{2x-3y}{3}+x+y=5, \\ \dfrac{4x+y}{5}=2. \end{cases}$

4. $\begin{cases} y+8=\dfrac{x}{2}, \\ x=\dfrac{y+56}{2}. \end{cases}$

5. $\begin{cases} \dfrac{x+2y}{3}+\dfrac{2y-3}{3}=2, \\ \dfrac{x}{3}+\dfrac{x+y}{3}=4. \end{cases}$

6. $\begin{cases} 3r+s=13, \\ \dfrac{2r-s}{5}+\dfrac{r+s}{7}=3. \end{cases}$

7. $\begin{cases} \dfrac{6}{x+y}=\dfrac{18}{x-y}, \\ \dfrac{10}{2x-3y}=\dfrac{30}{9y}. \end{cases}$

8. $\begin{cases} \dfrac{x}{7}=\dfrac{y}{6}, \\ \dfrac{x+y}{3}=x+y-26. \end{cases}$

9. $\begin{cases} 2x-\dfrac{y+2x}{7}=54, \\ \dfrac{2y-x}{7}-7=0. \end{cases}$

10. $\begin{cases} \dfrac{x+y}{3}-\dfrac{x+y}{2}=-5, \\ \dfrac{2x+3y}{7}+\dfrac{x+y}{5}=10. \end{cases}$

11. $\begin{cases} \dfrac{2x-y}{2}+\dfrac{x-2y}{2}=12, \\ \dfrac{x-3y}{2}-3x-2y=-28. \end{cases}$

12. $$\begin{cases} \dfrac{r-s+10}{2}-\dfrac{s-r-10}{2}=22, \\ \dfrac{r-s}{3}+\dfrac{r+s}{4}=14. \end{cases}$$

13. $$\begin{cases} \dfrac{3x+2y}{10}-\dfrac{2x-y}{2}=0, \\ x+y=28. \end{cases}$$

14. $$\begin{cases} \dfrac{r+s}{7}+\dfrac{r-s}{6}=\dfrac{46}{3}, \\ \dfrac{r-7s}{8}-\dfrac{r-6s}{14}=0. \end{cases}$$

For further exercises on this topic, see Appendix, p. 308.

131. Reciprocal Equations. The simultaneous equations thus far considered have been linear in x and y; that is, each equation has contained x and y to no higher power than the first (see § 123). Sometimes we have to solve equations which are linear, not in x and y, but in their reciprocals, $1/x$ and $1/y$. Such equations are called ***reciprocal equations*** and they may be solved without clearing of fractions, as appears from the following

EXAMPLE. Solve the equations

(1) $$\frac{1}{x}+\frac{2}{y}=2,$$

(2) $$\frac{2}{x}+\frac{3}{y}=\frac{7}{2}.$$

SOLUTION. Multiplying (1) by 3, and (2) by 2 and subtracting, we find

$$\frac{3}{x}-\frac{4}{x}=-1, \quad \text{or} \quad -\frac{1}{x}=-1, \quad \text{or} \quad \frac{1}{x}=1.$$

Therefore, $x=1$.

Substituting $x=1$ in (1) gives

$$1+\frac{2}{y}=2, \quad \text{or} \quad \frac{2}{y}=1.$$

Therefore, $y=2$.

The desired solution is therefore $(x=1,\ y=2)$. *Ans.*

EXERCISES

Solve each of the following pairs of simultaneous equations, and check your answers in Exs. 1–5.

1. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{4}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{4}. \end{cases}$

2. $\begin{cases} \dfrac{8}{x}-\dfrac{5}{y}=\dfrac{1}{6}, \\ \dfrac{7}{x}-\dfrac{3}{y}=\dfrac{5}{6}. \end{cases}$

[HINT. Multiply the first by 3 and the second by 5.]

3. $\begin{cases} \dfrac{5}{x}-\dfrac{3}{y}=-\dfrac{1}{6}, \\ \dfrac{3}{x}-\dfrac{1}{y}=\dfrac{1}{30}. \end{cases}$

4. $\begin{cases} \dfrac{5}{r}+\dfrac{10}{s}=\dfrac{9}{2}, \\ \dfrac{10}{r}-\dfrac{5}{s}=4. \end{cases}$

5. $\begin{cases} \dfrac{5}{y}-\dfrac{2}{z}=7, \\ \dfrac{2}{y}+\dfrac{2}{z}=0. \end{cases}$

6. $\begin{cases} \dfrac{2}{5s}+\dfrac{5}{2t}=7, \\ \dfrac{5}{s}+\dfrac{2}{t}=29. \end{cases}$

[HINT. Multiply the first by 4 and the second by 5.]

7. $\begin{cases} \dfrac{15}{2m}+\dfrac{1}{n}=2, \\ \dfrac{25}{2m}-\dfrac{3}{n}=1. \end{cases}$

8. $\begin{cases} \dfrac{5}{3p}+\dfrac{2}{5q}=7, \\ \dfrac{7}{6p}-\dfrac{1}{10q}=3. \end{cases}$

9. $\begin{cases} \dfrac{1}{x-1}+\dfrac{1}{y+1}=5, \\ \dfrac{2}{x-1}+\dfrac{3}{y+1}=12. \end{cases}$

[HINT. Solve as if $\dfrac{1}{x-1}$ and $\dfrac{1}{y+1}$ were the unknown numbers.]

10. $\begin{cases} \dfrac{5}{x-1}-\dfrac{3}{y-1}=14, \\ \dfrac{2}{x-1}-\dfrac{1}{y-1}=6. \end{cases}$

11. $\begin{cases} \dfrac{1}{x}=\dfrac{3}{2-y}, \\ \dfrac{5}{x}=\dfrac{6}{2-y}+9. \end{cases}$

EXERCISES — APPLIED PROBLEMS

1. Find two numbers whose sum is 32 and whose difference is 6.

[HINT. Let x represent one of the numbers, and y the other. Then, from the statement of the problem, we must have $x+y=32$ and $x-y=6$. Solve as in this Chapter, and check your answer.]

2. B had three times as much money as A. A then earned \$5 and B spent \$15, after which A had twice as much as B. How much did each have at first?

[HINT. Let $x=$ the number of dollars A had, and $y=$ the number of dollars B had. Then the two equations are $3x=y$ and $x+5=2(y-15)$. Explain.]

3. The sum of two numbers is 36, while their difference is 30. What are the numbers?

4. Find two numbers whose sum is -55 and whose difference is -5.

5. Find two numbers whose sum is 84 and such that the larger one is 48 more than three times the smaller one.

6. One half the sum of two numbers is 16, while one fourth their difference is -2. What are the numbers?

7. The sum of two numbers is zero, and their difference is 26. What are the numbers?

8. The perimeter (distance around) of a certain rectangle is 2 inches more than three times the base, and the base is $1\frac{4}{5}$ times the height. What are the base and height?

9. An isosceles triangle is one which has *two* of its sides equal to each other. If one of the three equal sides is 3 inches longer than the base, and the perimeter is 21 inches, what is the length of each of the three sides?

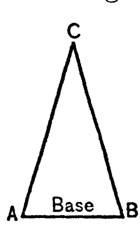


FIG. 72.

10. $ABCD$ is a parallelogram whose sides AB, BC, CD, and DA, are all equal to each other. The perimeter is 25 inches more than the altitude, and four times the altitude is 2 inches less than twice the base. Find the base and altitude.

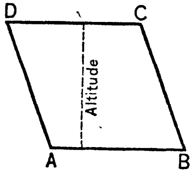


FIG. 73.

11. A father is twice as old as his son. Ten years ago he was three times as old as the son. Find the present age of each.

[HINT. Let x = the number of years in the father's age, and y = the number of years in the son's age. Then, their ages 10 years ago were $x-10$ and $y-10$.]

12. In 10 years B will be $\frac{3}{5}$ as old as A, and in 20 years B will be $\frac{2}{3}$ as old as A. Find the age of each.

13. A part of \$2500 is invested at 6% and the remainder at 5%. The yearly income from both is \$141. Find the amount in each investment.

14. A part of \$5000 is invested at 7% and the remainder at $3\frac{1}{2}$%. If the 7% investment brings each year \$193.75 more than the $3\frac{1}{2}$% investment, what is the amount of each?

15. A has a certain sum invested at 6% and B has another sum invested at the same rate. Their combined interest for one year is \$300, but it takes B 8 years to receive as much as A receives in one year. How much has each invested?

16. I have \$1.20 and I want to go to the moving picture exhibition as many times as possible. If I pay my trolley fares and entrance fees and ride both ways each time, I can go 6 times; if I walk one way every time I can go 8 times. What is the trolley fare and what is the price of admission?

17. A and B together can do a piece of work in 12 days. After A had worked alone for 5 days, B finished the work in 26 days. In what time can each alone do the work?

[HINT. Let x = the number of days in which A can do it alone, and y = the number of days in which B can do it alone. Then the part A can do in one day is $1/x$, and the part B can do in one day is $1/y$. (Compare Ex. 21, p. 171.) So the equations become

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{12}, \quad \text{and} \quad \frac{5}{x}+\frac{26}{y}=1. \quad \text{Explain.}$$

Now solve as in § 131.]

18. A and B can do a certain piece of work in 16 days. They work together for four days, when B is left alone and completes the work in 36 days. In what time could each do it separately?

19. If 4 boys and 6 men can do a piece of work in 30 days, and 5 boys and 5 men can do the same work in 32 days, how long will it take 12 men to do the work?

20. Two weights just balance on a lever 13 feet long when the fulcrum is 8 feet from one end. If their positions be reversed, it is necessary to add 78 pounds to the lesser weight to restore the balance. What are the weights?

[HINT. See § 112.]

PROBLEMS ON SPECIAL TOPICS

I. NUMBERS AND DIGITS

21. If 1 is added to the numerator of a certain fraction, the value of the fraction becomes $\frac{3}{4}$; if 2 is added to the denominator, the value becomes $\frac{1}{2}$. What is the fraction?

[HINT. Let x/y be the fraction.]

22. A certain fraction becomes equal to $\frac{4}{5}$ if $1\frac{3}{5}$ be added to both numerator and denominator, and it becomes equal to $\frac{1}{5}$ if $2\frac{1}{4}$ be subtracted from both numerator and denominator. What is the fraction?

23. A certain number of two digits equals four times the sum of the digits. What is the number? The digit in units' place is 3 greater than the digit in tens' place.

[HINT. Let x = the digit in tens' place, and y = the digit in units' place. Then the number itself is $10x + y$. (Why?)

24. The sum of the digits of a certain number of two figures is 5, and if three times the units' digit is added to the number, the order of the digits is reversed. What is the number?

II. Bills and Coins

25. An errand boy went to the bank to deposit some bills, some of them being \$1 bills and the rest \$2 bills. If there were 38 bills in all and their combined value was \$50, how many of each kind were there?

26. I have 15 coins, all silver dollars and quarters, whose value is \$9.00. How many of each denomination are there?

27. The receipts from the sale of 300 tickets for a musical recital were \$125. Adults paid 50 cents each, and children 25 cents each. How many tickets of each kind were sold?

III. Mixtures

28. A grocer wishes to make 50 pounds of coffee worth 32 cents per pound by mixing two other grades, one worth 26 cents per pound and the other 35 cents per pound. How much of each must he use?

[HINT. Let x = the amount to be used of the 26-cent grade, and y = the amount to be used of the 35-cent grade. Then $x + y = 50$ and $26x + 35y = 50 \times 32$. Why?]

29. A grain dealer wishes to sell feed consisting of a mixture of corn and oats for \$1.25 per hundred. Corn is worth \$1.30 per hundred and oats \$1.00. How many pounds of each must he put in each hundred of the mixture?

30. One cask contains 18 gallons of wine and 12 gallons of water; another, 4 gallons of wine and 12 of water. How many gallons must be taken from each cask so that when mixed there may be 21 gallons, half wine and half water?

IV. Problems on Motion

31. A and B run a race of 500 yards. In the first trial, A gives B a start of 80 yards and wins by 5 seconds. In the second trial, A gives B a start of 140 yards and B wins by 10 seconds. Find the rates of A and B in yards per second.

[Hint. Let x = A's rate, and y = B's rate. Then, the time it takes A to run the whole 500 yards is $500/x$ seconds (see § 107). In the first trial B runs but 420 yards and the time it takes him to do this is $420/y$. In the second trial B runs but 360 yards and the time it takes him to do this is $360/y$. Now form two equations which express the facts about the *times* as given in the problem.]

32. Two cities are 40 miles apart. To travel the distance between them by automobile takes 3 hours less time than by bicycle, but if the bicycle has a start of 24 miles, each takes the same time. What is the rate of the automobile, and what the rate of the bicycle?

33. A boy can row 10 miles downstream in 2 hours, and the same distance upstream in $3\frac{1}{3}$ hours. What is his rate of rowing in still water, and what is the rate of the stream?

[Hint. Let x = his rate of rowing in still water, and y = the rate of the stream. Then, his rate of rowing downstream is $x+y$, and his rate of rowing upstream is $x-y$.]

34. A boy rows 18 miles down a stream and back in 12 hours. He finds that he can row 3 miles downstream while he rows 1 mile upstream. What is his rate, and what is the rate of the stream?

35. A man rows for 4 hours down a stream which runs at the rate of 3 miles an hour. In returning it takes him $14\frac{1}{2}$ hours to reach a point 3 miles below his place of starting. Find the distance he rowed downstream and his rate in still water.

132. Literal Simultaneous Equations. Literal simultaneous equations are those in which letters are used to represent one or more of the *known* numbers. The letters used for this purpose are usually the first ones in the alphabet, a, b, c, etc. Such equations may be solved like those already considered in this Chapter, but it must be remembered that the solution must always be in terms of the known letters. (See § 106.)

EXAMPLE. Solve the equations

(1) $$ax+by=m,$$

(2) $$cx+dy=n.$$

SOLUTION.

(1)$\times d$ gives

(3) $$adx+bdy=dm.$$

(2)$\times b$ gives

(4) $$bcx+bdy=bn.$$

(3)$-$(4) gives $$(ad-bc)x=dm-bn$$

Hence $$x=\frac{dm-bn}{ad-bc}.\quad Ans.$$

(1)$\times c$ gives

(5) $$acx+bcy=cm.$$

(2)$\times a$ gives

(6) $$acx+ady=an.$$

(5)$-$(6) gives $$(bc-ad)y=cm-an$$

Hence $$y=\frac{cm-an}{bc-ad}.\quad Ans.$$

NOTE. In solving literal simultaneous equations it is usually best to perform all eliminations by addition or subtraction, thus avoiding substitution.

EXERCISES

Solve each of the following pairs of equations for x and y.

1. $\begin{cases} ax-by=m, \\ cx-dy=n. \end{cases}$

[See if your answer is correct when $a=2$, $b=1$, $c=3$, $d=4$, $m=2$, $n=1$, and thus check your work.]

2. $\begin{cases} ax+by=m, \\ bx-ay=n. \end{cases}$

3. $\begin{cases} ax=by, \\ x+y=ab. \end{cases}$

Check when $a=2$, $b=1$.

4. $\begin{cases} ax+3\,y=10\,a, \\ 4\,x-2\,y=6\,a. \end{cases}$

5. $\begin{cases} a(x-y)=5, \\ bx-cy=n. \end{cases}$

6. $\begin{cases} x-y=2\,n, \\ mx-ny=m^2+n^2. \end{cases}$

7. $\begin{cases} bx+ay=a+b, \\ ab(x-y)=a^2-b^2. \end{cases}$

8. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1, \\ \dfrac{x}{b}-\dfrac{y}{a}=\dfrac{1}{2}. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{b}. \end{cases}$

[HINT. See § 131.]

10. $\begin{cases} \dfrac{a}{x}-\dfrac{b}{y}=-1, \\ \dfrac{b}{x}-\dfrac{a}{y}=-1. \end{cases}$

11. The sum of two numbers is a, and their difference is b. Find the numbers.

12. One half the sum of two numbers is g, while one fourth their difference is h. What are the numbers?

13. I have c coins, all silver dollars and quarters, whose value is \$$d$. How many of each denomination are there? (Compare Exs. 25, 26, 27 p. 228.)

14. A grocer has two kinds of sugar, one worth a cents and the other b cents per pound. How many pounds of each must he take to make a mixture of c pounds worth d cents per pound? (Compare Exs. 28, 29, 30, p. 228.)

15. A and B run a race of a yards. In the first trial A gives B a start of b yards and wins by s seconds. In the second trial A gives B a start of c yards and B wins by t seconds. Find the rates of A and B in yards per second. (Compare Exs. 31, 32, 33, 34, p. 229.)

16. If A gives $\$d$ to B they have equal sums. If B gives $\$e$ to A, then A has 3 times as much as B. How much has each?

*** 133. Simultaneous Equations in Three Unknown Letters.** Sometimes we meet with three linear equations containing three unknown letters, instead of two equations in two letters. Such equations may be solved by the process of elimination, as will appear from a study of the following.

EXAMPLE. Solve the three simultaneous equations

(1) $$x+y+z=6,$$

(2) $$2x-y+3z=9,$$

(3) $$x+2y-z=2.$$

SOLUTION. Eliminate one unknown, say y, from (1) and (2). Thus,

(1)+(2) gives

(4) $$3x+4z=15.$$

Eliminate the same unknown, y, from (2) and (3), thus:

(2)×2 gives

(5) $$4x-2y+6z=18$$

From (3)

(6) $$x+2y-z=2$$

(5)+(6) gives $$5x+5z=20,$$

or

(7) $$x+z=4.$$

Equations (4) and (7) contain only x and z and hence may be solved for these letters, as in § 129. Thus,

from (4)

(8) $$3x+4z=15$$

(7)×3 gives

(9) $$3x+3z=12$$

(8)−(9) gives $$z=3.$$

Substituting $z=3$ in (7) gives

$$x+3=4.$$

Therefore $$x=1.$$

Substituting $z=3$ and $x=1$ in (1) gives

$$1+y+3=6.$$

Therefore, $$y=2.$$

The desired solution is therefore $x=1$, $y=2$, $z=3$. *Ans.*

CHECK. $1+2+3=6$, as required by (1).

$2\times 1-2+3\times 3=2-2+9=9$, as required by (2).

$1+2\times 2-3=1+4-3=2$, as required by (3).

An examination of the preceding solution gives the following rule.

RULE FOR SOLVING THREE SIMULTANEOUS EQUATIONS. *First eliminate one of the unknown letters from two of the given equations; then eliminate the same unknown from another two of the equations. This gives two new equations which contain only two of the unknown letters, and hence these two letters can then be found as in the solution of equations with two unknowns. The third letter can now be found by substitution of the two letters already obtained into one of the three original equations.*

EXERCISES

Solve for x, y, and z in the following and check your answers in the first four.

1. $\begin{cases} x+2y+3z=14, \\ 2x+y+2z=10, \\ 3x+4y-3z=2. \end{cases}$

2. $\begin{cases} x+3y-z=10, \\ 2x+5y+4z=57, \\ 3x-y+2z=15. \end{cases}$

3. $\begin{cases} x-y+z=30, \\ 3y-x-z=12, \\ 7z-y+2x=141. \end{cases}$

4. $\begin{cases} x+y=9, \\ y+z=7, \\ z+x=5. \end{cases}$

5. $\begin{cases} x+y-z=0, \\ x-y=2b, \\ x+z=3a+b. \end{cases}$

[HINT. See § 132.]

6. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=2, \\ \dfrac{2}{x}-\dfrac{1}{y}+\dfrac{1}{z}=7, \\ \dfrac{3}{x}+\dfrac{3}{y}+\dfrac{5}{z}=14, \end{cases}$

[HINT. Solve first for $\dfrac{1}{x}$, $\dfrac{1}{y}$ and $\dfrac{1}{z}$ as in § 131.]

7. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}-\dfrac{1}{z}=\dfrac{1}{a}, \\ \dfrac{1}{y}-\dfrac{1}{z}-\dfrac{1}{x}=\dfrac{1}{b}, \\ \dfrac{1}{z}-\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{c}. \end{cases}$

8. $\begin{cases} ax+by+cz=3, \\ x+y=\dfrac{a+b}{ab}, \\ y+z=\dfrac{b+c}{bc}. \end{cases}$

9. The sum of three numbers is 20. The sum of the first and second is 10 greater than the third, while the difference between the second and third is 6 less than the first. What are the numbers?

10. Three towns, A, B, and C, are connected by straight roads, which form a triangle. From A to B by way of C is 25 miles; from B to C by way of A is 30 miles; from C to A by way of B is 35 miles. How far apart are the towns?

11. A, B, and C have certain sums of money. B would have the same as A if A gave him \$100; C would have twice as much as A if A gave him \$100; and C would have four times as much as B if B gave him \$100. How much has each?

12. I have 35 coins, all nickels, dimes, and quarters, whose combined value is \$3.75, and there are half as many dimes as nickels. How many are there of each denomination?

13. A and B can do a piece of work in 10 days; A and C can do it in 8 days; and B and C can do it in 12 days. How long will it take each to do it alone?

[HINT. Let x, y, z = the number of days in which A, B, C, respectively, can do the work. Then find the part that each can do in one day. Compare Ex. 17, p. 227, and Ex. 21, p. 171.]

14. I have \$90 on deposit in the First National Bank, \$51 in the State Savings Bank, and \$75 in the Postal Savings Bank. If I have \$144 more to deposit, how shall I distribute it among the three banks so as to make all three deposits equal?

15. If I have \$$a$ in bank A, \$$b$ in bank B, and \$$c$ in bank C, how should I distribute \$$d$ more among them so as to equalize the accounts? Check your answer by seeing if it gives (in particular) the answer you have found for Ex. 14.

EXERCISES — REVIEW OF CHAPTERS XII–XIII

1. What is meant by the solution of a pair of simultaneous equations?

2. Draw the graphs of the following simultaneous equations, using the same axes for both. State any conclusions you thereby reach regarding their solution.

$$\begin{cases} x-y=-2, \\ 2x+y=-16. \end{cases}$$

3. If we have *three* simultaneous equations between the two letters x and y, and their three graphs all pass through one and the same point (point of intersection), then there will exist a certain value of x and a certain value of y that will satisfy all three equations at the same time. Why? Try this for the three equations

$$\begin{cases} x-2y=-3, \\ 3x-y=1, \\ x-y=-1. \end{cases}$$

4. In Ex. **3** what can be said if the three graphs do *not* all have the same point of intersection? Try this for the equations

$$\begin{cases} x-y=-1, \\ x-2y=-3, \\ 2x-3y=-5. \end{cases}$$

Solve each of the following pairs of equations by any method.

5. $\begin{cases} 2x+y=12, \\ x+y=3. \end{cases}$

6. $\begin{cases} 3x+y=16, \\ 3y+x=8. \end{cases}$

7. $\begin{cases} 5K+3T=8, \\ 7K-3T=4. \end{cases}$

8. $\begin{cases} 8w+9v=99, \\ 5w-7v=24. \end{cases}$

9. $\begin{cases} 2x-3y-8=0, \\ 7x-y-5=0. \end{cases}$

10. $\begin{cases} \dfrac{x+y}{4}-y=8, \\ \dfrac{x-y}{6}+y=0. \end{cases}$

11. $\begin{cases} \dfrac{r+s}{7}+\dfrac{r-s}{6}=\dfrac{46}{3}, \\ \dfrac{r-7s}{8}-\dfrac{r-6s}{14}=0. \end{cases}$

12. $\begin{cases} \dfrac{2x+3y+15}{7}=10, \\ \dfrac{x+y+8}{10}=3. \end{cases}$

13. Solve the reciprocal equations

$$\begin{cases} \dfrac{2}{3x}+\dfrac{3}{2y}=5, \\ \dfrac{3}{x}+\dfrac{2}{y}=13. \end{cases}$$

14. Solve by elimination

$$\begin{cases} \dfrac{x-1}{b-1}+\dfrac{y-a}{b-a}=1, \\ \dfrac{x+1}{b}+\dfrac{y-1}{1-a}=\dfrac{1}{b}. \end{cases}$$

* **15.** What is meant by the solution of three simultaneous equations between three unknown letters, x, y, and z? Describe how, in general, it is obtained.

* **16.** Solve for x, y, and z in the following

$$\begin{cases} x+y+2z=2(a+b), \\ x+z+2y=2(a+c), \\ y+z+2x=2(b+c). \end{cases}$$

* **17.** In a race of 500 yards, A can beat B by 20 yards, and C by 30 yards. By how much can B beat C in a good 500 yard race?

For further exercises on this chapter, see Appendix, pp. 307–308.

CHAPTER XIV

SQUARE ROOT

134. Definitions. The ***square root*** of a number is one of its two equal factors.

Thus, 2 is the square root of 4 because $2 \times 2 = 4$. Likewise, 3 is the square root of 9, etc.

The square root of a number is denoted by the ***radical sign*** $\sqrt{}$ placed over it.

Thus, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, etc.

To find the square root of a number is called *extracting* its square root.

135. Extracting Square Roots in Arithmetic. Many times we can pick out the square root of a number by inspection. Thus, $\sqrt{144}$ is seen to be 12, because $144 = 12 \times 12$. Similarly, $\sqrt{196} = 14$. But in finding such a root as $\sqrt{74529}$, we cannot get the answer by mere inspection. Here we may follow the process studied in arithmetic, which is indicated below and which should now be reviewed by the pupil.

PROCESS.

$$
\begin{array}{lr|l}
 & 7'45'29 & \underline{273} \quad \textit{Ans.} \\
 & 4 & \\
\text{Trial divisor} \quad = 2 \times 20 \ = 40 & 345 & \\
\text{Complete divisor} = 40 + 7 \ = 47 & \underline{329} & \\
\hline
\text{Trial divisor} \quad = 2 \times 270 = 540 & 1629 & \\
\text{Complete divisor} = 540 + 3 = 543 & \underline{1629} & \\
 & 0 &
\end{array}
$$

EXPLANATION. We first separate the number into periods of two figures each, beginning at the right. That is, we write it in the form

7′45′29. Find the greatest square in the left-hand period and write its root for the first figure of the required root. This gives the 2 appearing in the answer.

Square this root (giving 4), subtract the result from the left-hand period and to the remainder annex the next period for new dividend. This gives the 345 appearing in the process.

Double the root already found, with a cipher annexed (giving 40) for a trial divisor and divide the last dividend (345) by it. The quotient (or, in some cases, the quotient diminished) gives the second figure, 7, of the root.

Add to the trial divisor the figure last found (7), giving the complete divisor, 47. Multiply this complete divisor by the figure of the root last found (7), giving the 329 appearing in the process.

Subtract this from the dividend, and to the remainder annex the next period for the next dividend. This gives the 1629 of the process.

Proceed as before, and continue until a final new dividend equal to 0 is obtained. In the example above, this happens at once, giving 273 as the required root.

This process is usually studied in elementary arithmetic, and is stated here as a review of the arithmetic method. We shall see in § 136 that a similar process may be used in extracting square roots of algebraic expressions. A thorough mastery of the process for numbers will assist materially in understanding the algebraic case.

In the example just worked the root comes out *exactly* because 74529 is a *perfect square;* that is, it is like one of the numbers 1, 4, 9, 16, 25, 36, etc. If, on the other hand, we had started with a number which was *not* a perfect square, the process would not be different except that we should not finally reach a new dividend which was equal to 0. For such a number, in fact, the process goes on indefinitely, but if we stop it at any point we have the desired root correct (decimally) up to that point. For example, in getting the square root of 550 correct to two places of decimals, the process is as follows:

PROCESS.

$$
\begin{array}{r|l}
 & 5'50.'00'00 \,\big|\, \underline{23.45} \quad \textit{Ans.} \\
 & 4 \\
\hline
2\times 20 = 40 & 150 \\
40 + 3 = 43 & 129 \\
\hline
2\times 230 = 460 & 2100 \\
460 + 4 = 464 & 1856 \\
\hline
2\times 2340 = 4680 & 24400 \\
4680 + 5 = 4685 & \underline{23425} \\
 & \quad 975
\end{array}
$$

(Correct to two decimal places)

NOTE. In the process above we have first written 550 in the form 550.0000. If, instead of this, we had written it with *six* ciphers, that is, 550.000000, and then carried the process forward until all these were used below, we should have had the root correct to *three* places of decimals instead of two. In general, half the number of ciphers added in this way at the beginning is the number of decimal places to which the root obtained is correct.

The square roots of decimal numbers, like 334.796, are obtained like those for whole numbers except that in the beginning the separation of the number into periods of two figures each must be carried out *both* ways from the decimal point.

Thus, 334.796 would first be written as 3′34.′79′60. Similarly, 3.67893 would be written 3.′67′89′30. The extraction of the root is carried out from this point on just as shown in the process above.

EXERCISES

Find (by inspection or by the process shown in § 135) the square root of each of the following numbers.

1. 49.	**3.** 64.	**5.** 625.	**7.** 3844.
2. 81.	**4.** 225.	**6.** 529.	**8.** 2209.

9. 4761. **11.** 42025. **13.** $\frac{4}{9}$. **14.** $\frac{16}{25}$.

10. 57121. **12.** 95481. [HINT. $\frac{4}{9}=\frac{2}{3}\times\frac{2}{3}$.] **15.** $\frac{49}{81}$.

Find the square root correct to *two* decimal places, for each of the following numbers.

16. 382. **20.** 11. **23.** $\frac{3}{4}$.

17. 671. [HINT. Write as 11.'00'00] [HINT. Write as .75]

18. 1211. **21.** 7. **24.** $\frac{11}{12}$.

19. 12.96 **22.** 26. **25.** .08

136. Extracting Square Roots in Algebra. In extracting square roots in algebra we distinguish between several cases, according to the number of terms in the given expression.

Monomials. The square root of a monomial can usually be seen by inspection.

Thus,

$$\sqrt{36\ m^4n^2}=6\ m^2n,$$

because

$$6\ m^2n\times 6\ m^2n=36\ m^4n^2.$$

Similarly,

$$\sqrt{49\ x^6y^4z^8}=7\ x^3y^2z^4.$$

Trinomials. If a trinomial is a perfect square, its square root can be obtained by what has been said in §§ 63 and 64.

Thus, suppose we wish to find the square root of

$$9\ x^2+12\ xy+4\ y^2.$$

This trinomial is a perfect square, because its terms $9\ x^2$ and $4\ y^2$ are squares and positive, while its remaining term, $12\ xy$, is equal to $2\times 3\ x\times 2\ y$ (see § 63).

Being a perfect square, it can be expressed as the product of the two *equal* binomial factors $(3\ x+2\ y)(3\ x+2\ y)$, as explained in § 64. Whence, the desired square root of $9\ x^2+12\ xy+4\ y^2$ is $3\ x+2\ y$. *Ans.*

Similarly,

$$\sqrt{4\ s^2-4\ s+1}=2\ s-1, \text{ because } 4\ s^2-4\ s+1=(2\ s-1)(2\ s-1).$$

Polynomials. To find the square root of a polynomial (other than a trinomial) we use a process much like that followed in obtaining square roots in arithmetic. (See § 135.) This process is illustrated below.

EXAMPLE. Find the square root of

$$4x^4+12x^3-3x^2-18x+9.$$

PROCESS.

$$
\begin{array}{lll|l}
 & & 4x^4+12x^3-3x^2-18x+9 & \underline{2x^2+3x-3} \\
 & & \underline{4x^4} & \textit{Ans.} \\
\text{Trial divisor,} & 4x^2 & 12x^3-\ 3x^2 & \\
\text{Complete divisor,} & 4x^2+3x & \underline{12x^3+\ 9x^2} & \\
\text{Trial divisor,} & 4x^2+6x & -12x^2-18x+9 & \\
\text{Complete divisor,} & \underline{4x^2+6x-3} & \underline{-12x^2-18x+9} & \\
 & & 0 &
\end{array}
$$

EXPLANATION. We first arrange the terms of the polynomial in the descending (or ascending) powers of some letter. In the example, the arrangement is in descending powers of x.

Extract the square root of the first term, write the result as the first term of the root (giving the $2x^2$ in the answer), and subtract its square from the given polynomial (giving the $12x^3-3x^2$ in the second line of the process).

Divide the first term of the remainder by twice the root already found, used as a trial divisor, and the quotient, $3x$, is the next term in the root. Write this term in the root, and annex it to the trial divisor to form the complete divisor (giving the $4x^2+3x$).

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder (giving $-12x^2-18x+9$).

Find the next term of the root by dividing the first term of the remainder by the first term of the new trial divisor. This gives the -3 of the answer.

Form the complete divisor and continue in this manner until a remainder of zero is obtained.

EXERCISES

Find (by inspection or by the process shown in § 136) the square root of each of the following expressions. Square your answer in each case, and see if it gives (as it should) the given expression.

1. $9x^2y^2$.
2. $16a^4b^2$.
3. $36x^2y^2z^2$.
4. $81a^4b^8$.
5. $225x^6y^2$.
6. $625m^4n^6q^2$.
7. $196p^{10}q^{12}$.
8. $a^2b^4c^8d^6$.
9. $529r^8s^{12}$.
10. a^2x^{2m}.
11. $9m^2n^{4p}$.
12. $m^{2x}n^{4y}$.
13. x^2+2x+1.
14. x^2-2x+1.
15. $4a^2+12ab+9b^2$.
16. $4x^2+4xy+y^2$.
17. $c^2-4ac+4a^2$.
18. $9m^2-6mx+x^2$.
19. $x^2+xy+\frac{1}{4}y^2$.
20. $4x^4-52x^2+169$.
21. $(a+b)^2-4(a+b)+4$.
22. $x^4+2x^3+3x^2+2x+1$.
23. $9x^4-12x^3+10x^2-4x+1$.
24. $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.
25. $x^4-6x^3y+13x^2y^2-12xy^3+4y^4$.

[HINT. This Example differs slightly from the one worked in § 136 because we here have *two* letters, x and y, in the polynomial. The process, however, remains as before.]

26. $x^8+2a^6x^2-a^4x^4-2a^2x^6+a^8$.

[HINT. First arrange in descending powers of x. That is, $x^8-2a^2x^6-a^4x^4$ . . .]

27. $9x^2-6xy+y^2+12xz-4yz+4z^2$.
28. $9x^2+25y^2+9z^2-30xy+18xz-30yz$.
29. $x^8+4x^7+68x^3-20x^5-2x^6+15x^4-50x^2-88x+121$.
30. Find the first four terms of the square root of $1+x$.

SOLUTION.

$$
\begin{array}{r|l}
 & 1+x \quad \underline{\left|\, 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3 \right.} \quad \textit{Ans.} \\
 & \underline{1} \qquad\qquad (\textit{Correct to four terms}) \\
 & x \\
2+\tfrac{1}{2}x & \underline{x+\tfrac{1}{4}x^2} \\
 & -\tfrac{1}{4}x^2 \\
2+x-\tfrac{1}{8}x^2 & \underline{-\tfrac{1}{4}x^2-\tfrac{1}{8}x^3+\tfrac{1}{64}x^4} \\
2+x-\tfrac{1}{4}x^2+\tfrac{1}{16}x^3 & \tfrac{1}{8}x^3-\tfrac{1}{64}x^4
\end{array}
$$

NOTE. We are here extracting the root of an expression, $1+x$, which is *not* a perfect square. As with arithmetic numbers that are not perfect squares, the process leads to an answer which is correct up to the point where it is stopped and which affords an *approximation* that approaches more closely the exact result the farther the process is continued. Compare § 135.

31. Find (as in Ex. 30) the first four terms of the square root of the following expressions:

(*a*) $1-x$. (*b*) x^2+1. (*c*) x^2-1.

For further exercises on this topic, see Appendix, p. 308.

137. The Double Sign of the Square Root. We know that 3 is the square root of 9 because $3\times 3=9$. But we also have $(-3)\times(-3)=9$. Therefore, -3 can also be regarded as a square root of 9. In other words, 9 has *two* square roots, $+3$ and -3, which are opposite in sign, but otherwise the same. Similarly, 16 has the two square roots, $+4$ and -4. In general, a^2 has the two square roots a and $-a$.

The double sign $\pm$ is sometimes used. Thus, we say that the square root of 9 is ± 3. This is merely a way of saying briefly that the two roots are $+3$ and -3.

In order to avoid all confusion, it is to be understood that the radical sign $\sqrt{}$ when placed over a number means the *positive* square root of that number. If it is desired to

indicate the negative square root, it is done by the symbol $-\sqrt{}$.

Thus, $\sqrt{16}$ means $+4$, while $-\sqrt{16}$ means -4. Similarly, $\sqrt{a^2}=a$ and $-\sqrt{a^2}=-a$.

138. Equations Containing Radical Signs. Equations containing radical signs are often solved by squaring each member. This is equivalent to multiplying each member by the same quantity, and hence is justified by Axiom III (§ 9).

EXAMPLE 1. Solve the equation $\sqrt{x-2}=6$.

SOLUTION. Squaring both members gives

$$x-2=36.$$

Whence, $x=38$. *Ans.*

CHECK. $\sqrt{38-2}=\sqrt{36}=6$.

EXAMPLE 2. Solve the equation $\sqrt{x-1}-\sqrt{x-4}=1$.

SOLUTION. First, transpose the $-\sqrt{x-4}$ to the right, giving

$$\sqrt{x-1}=1+\sqrt{x-4}.$$

Square both members, using Formula V, p. 101, for finding $(1+\sqrt{x-4})^2$. This gives

$$x-1=1+2\cdot 1\cdot\sqrt{x-4}+(\sqrt{x-4})^2,$$

or, $$x-1=1+2\sqrt{x-4}+x-4.$$

Canceling x from both sides and transposing the 1 and -4 to the left side, gives

$$2=2\sqrt{x-4}, \quad \text{or} \quad \sqrt{x-4}=1.$$

Whence (squaring again),

$$x-4=1^2=1.$$

Therefore $x=5$. *Ans.*

CHECK. $\sqrt{5-1}-\sqrt{5-4}=\sqrt{4}-\sqrt{1}=2-1=1$.

NOTE. It is especially important to check all answers as above for equations containing radicals, since the process of squaring both members sometimes leads to a new equation whose roots do not *all* belong to the first one. Thus, if we square both members of the equation $x=5$ we get $x^2=25$, and this last equation has $x=-5$ as a root as well as $x=5$.

EXERCISES

Solve each of the following equations, and check your answer in each case.

1. $\sqrt{x-10}=5.$
2. $4=\sqrt{x-3}.$
3. $\sqrt{2x+1}=3.$
4. $\sqrt{x^2+5}=3.$
5. $\sqrt{x+5}-\sqrt{x}=1.$
6. $\sqrt{x-2}+\sqrt{x+3}=5.$
7. $\sqrt{2x+5}-\sqrt{2x+2}=1.$
8. $\sqrt{3x+7}-\sqrt{2x+10}=0.$
9. $\sqrt{x+a}=\sqrt{x}+a.$

[HINT. This being a literal equation (see § 106) we are to find x *in terms of* a.]

10. $\sqrt{2x-3a^2}+\sqrt{2x}=3a.$
11. $\dfrac{\sqrt{x+8}}{\sqrt{2x+7}}=\dfrac{\sqrt{x+3}}{\sqrt{2x+2}}.$

[HINT. Square both sides, remembering that the square of a fraction equals the square of its numerator divided by the square of its denominator.]

12. $\dfrac{\sqrt{a-3x}}{2\sqrt{x}}=\dfrac{2\sqrt{x}}{\sqrt{a+3x}}.$

13. If 10 be added to 2 times a certain number, the square root of the result is 4. What is the number?

[HINT. Form an equation and solve it.]

14. If 9 be added to the square of a certain number, the square root of the result is 5. What is the number?

15. The difference between the square root of a certain number and the square root of 11 less than that number is 1. What is the number?

For further exercises on this chapter, see Appendix, pp. 308–309.

CHAPTER XV

RADICALS

139. Radicals. Suppose we have a square (see Fig. 74) which we know contains exactly 2 sq. ft. How long is each of its four sides? In order to answer this, we let x represent the desired length. Then, we must have

$$x \cdot x = 2,$$

or,

$$x^2 = 2.$$

Hence,

$$x = \sqrt{2} \text{ ft.} \quad \textit{Ans.}$$

This number, $\sqrt{2}$, cannot be found *exactly*, as explained in § 135, because it is the square root of a number, 2, which is not a perfect square. Still, $\sqrt{2}$ measures a perfectly definite length, as indicated in Fig. 74. The value of it, *correct to two decimal places only*, is, by the process in § 135, found to be 1.41.

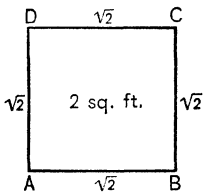


FIG. 74.

Such a number as $\sqrt{2}$ is called a ***radical number***, or briefly, a ***radical***. It is an indicated root whose value cannot, in general, be *exactly* determined.

LEIBNITZ

(*Gottfried Wilhelm von Leibnitz*, 1646–1716)

Celebrated for his interest and ability in all branches of learning, especially in the fields of mathematics and philosophy. A contemporary of Newton and regarded as sharing with him the honor of inventing the Calculus.

The word *radical* is used in connection with other roots than square roots. Thus, $\sqrt[3]{10}$ means the *cube* root of 10; that is, the number which when used as a factor *three* times gives 10. Similarly, $\sqrt[4]{6}$ means the *fourth* root of 6, etc. All such numbers represent perfectly definite magnitudes, as did $\sqrt{2}$ in Fig. 74, yet we cannot express them *exactly* by means of decimals.

In general, the nth root of any number a is written $\sqrt[n]{a}$, and this is known as a radical of the nth order. The number n is here called the ***index*** of the root, and the number a itself is called the ***radicand***.

When no index is expressed, the index 2 is understood. Thus, $\sqrt{3}$ means $\sqrt[2]{3}$.

These definitions apply also to algebraic expressions. Thus, $\sqrt{3xy^2}$ and $\sqrt{9x^2y^2}$ are both radicals, although $9x^2y^2$ is a perfect square, thus making it possible to write $\sqrt{9x^2y^2} = 3xy$.

EXERCISES

In the following list, pick out those values that can be expressed exactly without radicals, and those that cannot. For each state the index and the radicand.

1. $\sqrt{7}$.

2. $\sqrt{9}$.

3. $\sqrt{17}$.

4. $\sqrt{\frac{8}{7}}$.

5. $\sqrt{\frac{4}{9}}$.

6. $\sqrt{.3}$

7. $\sqrt[3]{-8}$.

[HINT. — $8 = (-2)^3$.]

8. $\sqrt[3]{11}$.

9. $\sqrt[3]{27}$.

10. $\sqrt[4]{5}$.

11. $\sqrt[4]{16}$.

12. $\sqrt[5]{50}$.

13. $\sqrt[5]{32}$.

14. $\sqrt{3x^2y^2}$.

15. $\sqrt{64a^6b^4}$.

16. $\sqrt[3]{8a^6b^3}$.

17. $\sqrt[4]{5x^6y^2}$.

140. Value of Radicals. Use of Tables. To get the value of a radical correct to two or more places of decimals usually calls for a rather long process, as we have seen in § 135,

where we found the value of 550. For this reason, the radicals which are needed most in ordinary life (that is, those coming from square roots and cube roots) have been carefully worked out and placed together in a Table at the end of this book, on p. 314. For the sake of completeness, the squares and cubes of the numbers are also shown, thus making the table very convenient for all kinds of mathematical work. Just how to use the table is described on page 311, which the pupil should now read carefully. Below are a few illustrative examples:

EXAMPLE 1. Find $\sqrt{7}$ from the table.

SOLUTION. By the top number in the *third* column of p. 326 (Table) we see that $\sqrt{7}=2.64575$, this value being correct to 5 decimal places.

EXAMPLE 2. Find $\sqrt[3]{7}$ from the table.

SOLUTION. By the top number in the *sixth* column of p. 326 we have $\sqrt[3]{7}=1.91293$, this value being correct to 5 decimal places.

EXAMPLE 3. Find $\sqrt{70}$.

SOLUTION. By the top number in the *fourth* column of p. 326 we have $\sqrt{70}=8.36660^{+}$. The sign $+$ which we have placed over the last digit indicates that the number is correct only up to the last decimal place.

EXAMPLE 4. Find $\sqrt[3]{70}$.

SOLUTION. $\sqrt[3]{70}=4.12129^{+}$ from 7th column top of p. 326.

EXAMPLE 5. Find $\sqrt[3]{700}$.

SOLUTION. $\sqrt[3]{700}=8.87904^{+}$ by top of 8th column, p. 326.

141. Artisan's Method for Finding Square Root. The square root of a number may be easily calculated correct to several places of decimals in case one knows its value correct merely to the first or second decimal place. For example, knowing that $\sqrt{2}=1.41^{+}$, suppose we wish to calculate $\sqrt{2}$ to a greater degree of accuracy. We first

divide the 2 by the 1.41, carrying out the work to several places of decimals, as indicated below.

$$\begin{array}{r|l}
1.41 & 2. \qquad \underline{|1.41851} \\
 & \underline{1.41} \\
 & 590 \\
 & \underline{564} \\
 & 260 \\
 & 141 \\
 & 1190 \\
 & 1128 \\
 & 720 \\
 & 705 \\
 & 150 \\
 & \underline{141}
\end{array}$$

Now, since 1.41 was known to be slightly *less* than the true root, it follows that our quotient, 1.41851, is slightly *greater* than the true root. In other words, the true root itself lies somewhere between these two numbers. So we now calculate the number which lies halfway between them (their *average*) by taking half their sum. Thus,

$$\begin{array}{r|l}
 & 1.41 \\
 & \underline{1.41851} \\
2 & \underline{2.82851} \\
 & 1.41426
\end{array}$$

This average, or 1.41426, gives us a very close approximation to the true root. In fact, our answer in the present instance is the correct value of $\sqrt{2}$ to *four* places of decimals.†

† In general, if the number whose square root is to be found is greater than one, the new result will be correct at least to *twice as many decimal places* as the first estimate. The first estimate need not be at all exact, however. If any arbitrary number is taken as the first estimate, the process will give accurate answers after a sufficient number of repetitions.

Similarly, this method may be used to secure a good approximation for the square root of any number in case the value of the root is already known correct to only one or two decimal places. If still greater accuracy is desired, simply repeat the same process.

Because of the convenience and simplicity of this process, especially in rapid calculations such as are made by engineers, mechanics, or other workmen, it is commonly known as the "*artisan's method*."

EXERCISES

Using the tables, find the approximate values of each of the following quantities.

1. $\sqrt{5}$. **2.** $\sqrt{50}$. **3.** $\sqrt[3]{5}$. **4.** $\sqrt[3]{50}$. **5.** $\sqrt[3]{500}$.

6. $\sqrt{51}$.

[HINT. $51=5.1\times 10$. So use the information given for 5.10 on page 322 of Table.]

7. $\sqrt[3]{51}$. **8.** $\sqrt{82}$. **9.** $\sqrt[3]{820}$. **10.** $\sqrt{7.7}$ **11.** $\sqrt{7.71}$

12. $\sqrt{77.1}$ **13.** $\sqrt{771}$.

SOLUTION. $771=7.71\times 100$. Therefore $\sqrt{771}$ is the same as $\sqrt{7.71}$ except that the decimal point in the root must be moved one place farther to the right. (See p. 311.) Now, $\sqrt{7.71}=2.77669^+$, so it follows that $\sqrt{771}=27.7669^+$. *Ans.*

14. $\sqrt{552}$.

[HINT. See solution of Ex. 13.]

15. $\sqrt{367}$. **16.** $\sqrt{984}$.

17. $\sqrt[3]{7110}$.

[HINT. $7110=7.11\times 1000$. See Ex. 13.]

18. $\sqrt[3]{6560}$.

19. $\sqrt{.55}$

[HINT. $.55=\frac{1}{10}\times 5.5$. Now see p. 312.]

20. $\sqrt{.0676}$

EXERCISES — APPLIED PROBLEMS

In answering the following, use the tables on pp. 314–331.

1. Figure 75 represents a right-angled triangle, whose short sides AB and BC measure 3 inches and 2 inches respectively. What is the length of the long side AC (hypotenuse)?

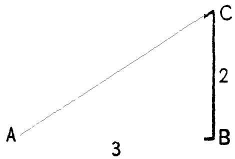


Fig. 75.

Solution. Let x represent the length of AC. Then $x^2 = 3^2 + 2^2 = 9 + 4 = 13$. Therefore, $x = \sqrt{13} = 3.60555^+$ inches (Table). *Ans.*

2. A baseball diamond is a square 90 ft. on a side. What is the distance from home plate to second base?

3. The diagonal of a square is 12 feet long. What is the length of each side?

4. The dimensions of a certain rectangular field are 100 feet by 230 feet. In going from one corner to the opposite corner how much shorter is it to go by the diagonal than to go around?

5. If the area of a circle is 10 square inches, how long is the radius?

[Hint. See Ex. 25, p. 22. Take $\pi = 3\frac{1}{7}$.]

6. If the volume of a cube is 235 cubic inches, how long is the edge?

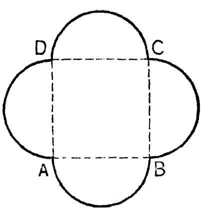


Fig. 76.

7. If the volume of a sphere is 672 cubic inches, how long is the radius?

[Hint. See Ex. 28, p. 23. Take $\pi = 3\frac{1}{7}$.]

8. In the accompanying figure, how long should the radius of each semicircle be in order that the entire area inclosed may be 200 square feet?

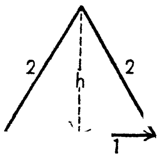


Fig. 77.

9. Figure 77 represents a triangle each of whose three sides is 2 inches long. What is the area?

[Hint. Here it is necessary to find the altitude h. Note that it is a side of a right triangle whose hypotenuse is 2 inches and whose other side is 1 inch.]

10. Find the ratio between the radius of a sphere and the radius of a second sphere whose volume is 7 times that of the first.

[Hint. The volumes of any spheres are to each other as the cubes of their radii. See § 115.]

142. Simplification of Radicals. We know that the square root of the product of two numbers is the same as the product of their square roots. For example, $\sqrt{4\times 25}$ is the same as $\sqrt{4}\times\sqrt{25}$ because both are equal to 10. (Explain.) In the same way, $\sqrt[3]{8\times 3}$ may be written $\sqrt[3]{8}\times\sqrt[3]{3}$, or simply $2\sqrt[3]{3}$. In fact, we have the following general formula:

Formula X. $$\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}.$$

Again, $\sqrt{\frac{4}{9}}$ is the same as $\dfrac{\sqrt{4}}{\sqrt{9}}$ because both are equal to $\frac{2}{3}$. (Explain.) Similarly, $\sqrt[3]{\frac{5}{8}}$ may be written $\dfrac{\sqrt[3]{5}}{\sqrt[3]{8}}$, or $\dfrac{\sqrt[3]{5}}{2}$. So in general we have the following formula:

Formula XI. $$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Formulas X and XI enable us to simplify many radical expressions, as is illustrated by the following examples:

Example 1. Simplify $\sqrt{63}$.

Solution. $\sqrt{63}=\sqrt{9\cdot 7}=\sqrt{9}\cdot\sqrt{7}=3\sqrt{7}$. *Ans.*

Example 2. Simplify $\sqrt[3]{32}$.

Solution. $\sqrt[3]{32}=\sqrt[3]{8\cdot 4}=\sqrt[3]{8}\cdot\sqrt[3]{4}=2\sqrt[3]{4}$. *Ans.*

Example 3. Simplify $\sqrt{\frac{8}{27}}$.

Solution. $$\sqrt{\frac{8}{27}}=\frac{\sqrt{8}}{\sqrt{27}}=\frac{\sqrt{4\cdot 2}}{\sqrt{9\cdot 3}}=\frac{\sqrt{4}\cdot\sqrt{2}}{\sqrt{9}\cdot\sqrt{3}}=\frac{2\sqrt{2}}{3\sqrt{3}}. \quad \textit{Ans.}$$

Example 4. Simplify $\sqrt{20\,a^6}$.

Solution. $\sqrt{20\,a^6}=\sqrt{4\,a^6\cdot 5}=\sqrt{4\,a^6}\cdot\sqrt{5}=2\,a^3\sqrt{5}.$ *Ans.*

Example 5. Simplify $\sqrt[3]{\dfrac{72\,x^2y^6}{z^6}}$.

Solution.

$$\sqrt[3]{\frac{72\,x^2y^6}{z^6}}=\frac{\sqrt[3]{(8\,y^6)(9\,x^2)}}{\sqrt[3]{z^6}}=\frac{\sqrt[3]{8\,y^6}\cdot\sqrt[3]{9\,x^2}}{z^2}=\frac{2\,y^2\sqrt[3]{9\,x^2}}{z^2}. \quad \textit{Ans.}$$

143. Note. It should be observed that in each of the examples above the process of simplification consists in removing from under the radical sign the largest factor of the radicand that is a perfect square, perfect cube, etc. Thus, in Example 1, the radicand, 63, is first broken up into factors in such a way that 9 (which is a perfect square) appears clearly. Similarly, in Example 2 (where we are dealing with a *cube* root) we first write the radicand, 32, in a form which brings out to the eye its factor 8, which is a perfect cube. The first step in all such examples is, therefore, to get the radicand broken up into factors. This requires good judgment, but becomes very easy after slight practice and experience. The pupil is especially warned that one *cannot* write $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$. Thus, when $a=4$, $b=9$ this would give $\sqrt{13}=2+3$, which is clearly *false*.

EXERCISES

1. By Formula X we may write $\sqrt{20}=\sqrt{4\cdot 5}=2\sqrt{5}$. By looking up $\sqrt{20}$ and $\sqrt{5}$ in the tables, show that $\sqrt{20}$ *is* the same as $2\sqrt{5}$.

2. Show by the tables (as in Ex. 1) that $\sqrt{54}=3\sqrt{6}$.

Simplify each of the following expressions. (See § 143.)

3. $\sqrt{45}$. **4.** $\sqrt{80}$. **5.** $\sqrt{288}$. **6.** $\sqrt{\frac{75}{12}}$.

7. $\sqrt[3]{\frac{24}{135}}$. **8.** $\sqrt{98}$. **9.** $\sqrt{16\,a^5b^3}$.

10. $\sqrt{\dfrac{a^3y}{b^4z^2}}$.

11. $\sqrt{\dfrac{16\,mn^4}{s^3t}}$.

12. $\sqrt{\dfrac{27\,a^3b^3c^7}{8\,xyz}}$.

13. $\sqrt{\dfrac{3(a+b)^2c^2d}{4(a^2-b^2)}}$.

14. $\sqrt[5]{\dfrac{64\,a^7m^6n^8}{243\,xyz}}$.

15. $\sqrt{\dfrac{32\,x^3(y-z)^2}{m^4n^3}}$.

Put each of the following in a form *without* a number written outside the radical sign.

16. $2\sqrt{3}$.

SOLUTION. $2\sqrt{3}=\sqrt{4}\cdot\sqrt{3}=\sqrt{12}$. *Ans.* (Formula X.)

17. $3\sqrt{2}$.

18. $\frac{1}{2}\sqrt{3}$.

[HINT. Write as $\dfrac{\sqrt{3}}{\sqrt{4}}$ and apply Formula XI.]

19. $\frac{1}{4}\sqrt{7}$.

20. $\frac{1}{2}\sqrt{8}$.

21. $\dfrac{a}{b}\sqrt{\dfrac{b}{a}}$.

22. $2\,x\sqrt{3\,y}$.

23. $ab\sqrt{3\,cd}$.

24. $3\,ab\sqrt{cd}$.

25. $2\,x\sqrt{a^2-b^2}$.

26. $m\sqrt[3]{n}$.

27. $2\,a\sqrt[3]{a-b}$.

28. $\dfrac{a\sqrt{x^3}}{\quad}$.

29. $\frac{2}{3}\sqrt{2}$.

30. $\frac{2}{3}\sqrt{\frac{4}{25}}$.

31. $(a-b)\sqrt{2\,c}$.

32. $\dfrac{m}{n}\sqrt{a^2-b^2}$.

33. $\dfrac{2\,c}{d}\sqrt[3]{1-x}$.

34. $\dfrac{a+b}{m}\sqrt{\dfrac{m^3}{a^2-b^2}}$

144. Similar Radicals. Addition and Subtraction of Radicals. Whenever two radicals with the same index have the *same* radicand, or can be given the same radicand by simplification, they are called ***similar radicals***.

Thus, $2\sqrt{2}$ and $3\sqrt{2}$ are similar radicals; so also are $\sqrt{2}$ and $\sqrt{32}$, since the last of these may be simplified into $4\sqrt{2}$. Likewise, $\sqrt{3\,a^2x}$ and $\sqrt{3\,b^2x}$ are similar, being equal respectively to $a\sqrt{3\,x}$ and $b\sqrt{3\,x}$.

Whenever similar radicals are added or subtracted the result may be expressed in a single term.

Thus, $4\sqrt{3}+5\sqrt{3}=(4+5)\sqrt{3}=9\sqrt{3}$. *Ans.*

Again, $3\sqrt{32}-2\sqrt{8}=3\cdot4\sqrt{2}-2\cdot2\sqrt{2}=12\sqrt{2}-4\sqrt{2}$
$=(12-4)\sqrt{2}=8\sqrt{2}$. *Ans.*

Likewise, $2\sqrt{4\,a^2b}+\sqrt{9\,a^2b}-\sqrt{16\,a^2b}=2\cdot 2\,a\sqrt{b}+3\,a\sqrt{b}-4\,a\sqrt{b}$
$=(4\,a+3\,a-4\,a)\sqrt{b}$
$=3\,a\sqrt{b}.$ *Ans.*

EXERCISES

Combine the radicals in the following exercises whenever possible. Check your answer in Exs. 1–4 by use of the tables.

1. $\sqrt{8}+\sqrt{18}+\sqrt{32}.$

2. $\sqrt{12}+\sqrt{27}-\sqrt{75}.$

3. $\sqrt[3]{2}+\sqrt[3]{16}-\sqrt[3]{54}.$

4. $\sqrt{128}-\sqrt{18}+\sqrt{72}.$

5. $\sqrt{\frac{3}{4}}+\sqrt{\frac{27}{16}}+\sqrt{\frac{75}{9}}.$

6 $\sqrt{8}+\sqrt{12}+\sqrt{16}.$

7. $\sqrt{32\,a^2}-\sqrt{8\,a^2}+\sqrt{18\,a^2}.$

8. $\sqrt[3]{16\,a^3b^3}+\sqrt[3]{54a^3b^3}.$

9. $\sqrt{32\,a}-\sqrt{8\,a}+\sqrt{18\,a}.$

10. $\sqrt{2(x-y)^2}+\sqrt{8(x-y)^2}+\sqrt{18(x-y)^2}.$

11. $\sqrt{2(x-y)}+\sqrt{8(x-y)}+\sqrt{18(x-y)}.$

12. $\sqrt{\frac{1}{2}}+\sqrt{12\frac{1}{2}}+\sqrt{\frac{1}{8}}+\sqrt{1\frac{1}{8}}.$

13. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+3\sqrt{27}.$

14. $\sqrt[3]{\frac{8}{5}}+\sqrt[3]{\frac{1}{5}}+\sqrt[3]{5\frac{2}{5}}.$

15. $\sqrt{\frac{a}{x^2}}+\sqrt{\frac{a}{y^2}}-\sqrt{\frac{a}{z^2}}.$

16. $\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ac}}+\sqrt{\frac{c}{ab}}.$

145. Definition of Surds. A square root which cannot be extracted exactly is commonly called a ***quadratic surd***, or more briefly, a ***surd***.

A surd is thus merely a radical whose index is 2 (see § 139) whose value cannot be expressed exactly. For example, $\sqrt{3}$, $\sqrt{\frac{5}{6}}$, $\sqrt{6.5}$, $\sqrt{a+b}$, $\sqrt{x^2-y^2}$ are surds, but $\sqrt{9}$, $\sqrt{25}$, $\sqrt{x^2+2\,xy+y^2}$ are not.

146. Multiplication of Surds. To multiply one surd, $\sqrt{a}$, by another, $\sqrt{b}$, we have the principle

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}.$$

This principle is, in fact, what Formula X gives when $n=2$.

EXAMPLE 1. Find the product of $\sqrt{2}$ and $\sqrt{18}$.

SOLUTION. $\sqrt{2}\cdot\sqrt{18}=\sqrt{2\cdot 18}=\sqrt{36}=6.$ *Ans.*

EXAMPLE 2. Multiply $\sqrt{3}+\sqrt{5}$ by $2\sqrt{3}-\sqrt{5}$.

SOLUTION.

$$\begin{array}{l} \sqrt{3}+\sqrt{5} \\ 2\sqrt{3}-\sqrt{5} \\ \hline 2\cdot 3+2\sqrt{15} \\ \quad -\sqrt{15}-5 \\ \hline 6+\sqrt{15}-5=1+\sqrt{15}. \quad \textit{Ans.} \end{array}$$

EXAMPLE 3. Multiply $\sqrt{a}+\sqrt{a-b}$ by $\sqrt{a}-\sqrt{a-b}$.

SOLUTION.

$$\begin{array}{l} \sqrt{a}+\sqrt{a-b} \\ \sqrt{a}-\sqrt{a-b} \\ \hline a+\sqrt{a^2-ab} \\ \quad -\sqrt{a^2-ab}-(a-b) \\ \hline a+0 \qquad -(a-b)=a-a+b=b. \quad \textit{Ans.} \end{array}$$

EXERCISES

Simplify each of the following expressions as far as possible.

1. $\sqrt{3}\cdot\sqrt{27}$.
2. $\sqrt{8}\cdot\sqrt{12}$.
3. $\sqrt{6}\cdot\sqrt{4}$.
4. $\sqrt{5}\cdot\sqrt{6}$.
5. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{8}}$.
6. $\sqrt{5}\cdot\sqrt{\frac{1}{5}}$.
7. $\sqrt{.1}\cdot\sqrt{.001}$
8. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$.
9. $(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})$.
10. $(\sqrt{5}-\sqrt{2})^2$.
11. $(\sqrt{5}-2)^2$.
12. $(2\sqrt{5}+3\sqrt{3})(4\sqrt{5}-5\sqrt{3})$.
13. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3})$.
14. $\sqrt{x}\cdot\sqrt{x^3}$.
15. $\sqrt{ab}\cdot\sqrt{a^3b^3}$.
16. $\sqrt{x^2y}\cdot\sqrt{xy}$.
17. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.
18. $(\sqrt{a}+\sqrt{b})^2$.
19. $(\sqrt{3\,x}+\sqrt{4\,y})^2$.
20. $(3\sqrt{x}+4\sqrt{y})(2\sqrt{x}-5\sqrt{y})$.
21. Find the value of x^2 if $x=\sqrt{3}+\sqrt{2}$.
22. Find the value of $x^2-4\,x-1$ if $x=2+\sqrt{5}$.
23. Find the value of $x^2+3\,x-2$ if $x=(\sqrt{17}-3)/2$.
24. Does $\sqrt{3}+\sqrt{2}$ "satisfy" the equation $x^2-4\,x+1=0$; that is, is this equation true when $x=\sqrt{3}+\sqrt{2}$?

147. Division of Surds. To divide one surd, $\sqrt{a}$, by another, $\sqrt{b}$, we have the principle

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}.$$

This principle is simply what Formula XI gives when $n=2$.

EXERCISES

Express each of the following quotients as a fraction under one radical sign, and reduce to simplest form.

1. $\dfrac{\sqrt{15}}{\sqrt{3}}$.

SOLUTION. $\dfrac{\sqrt{15}}{\sqrt{3}}=\sqrt{\dfrac{15}{3}}=\sqrt{5}.$ *Ans.*

2. $\dfrac{\sqrt{24}}{\sqrt{6}}$.
3. $\dfrac{\sqrt{2}}{\sqrt{8}}$.
4. $\dfrac{\sqrt{3}}{\sqrt{5}}$.
5. $\dfrac{1}{\sqrt{2}}$.
6. $\dfrac{\sqrt{x^3}}{\sqrt{x^5}}$.
7. $\dfrac{\sqrt{a^2-b^2}}{\sqrt{a-b}}$.
8. $\dfrac{\sqrt{3(x^2-y^2)}}{\sqrt{12(x+y)^3}}$.

148. Rationalizing the Denominator of a Fraction. If the denominator of a fraction consists of a single surd, or is a binomial containing surds, the fraction may be changed into one which has surds only in its numerator. The process of doing this is called ***rationalizing the denominator***, and is illustrated below.

EXAMPLE 1. Rationalize the denominator in the fraction

$$\frac{\sqrt{3}}{\sqrt{5}}.$$

SOLUTION. Here the denominator consists of the single surd $\sqrt{5}$. To get this surd out of the denominator it is merely necessary to multiply both numerator and denominator of the fraction by $\sqrt{5}$, giving $\sqrt{15}/5$. *Ans.*

EXAMPLE 2. Rationalize the denominator in the fraction

$$\frac{1}{\sqrt{3}+\sqrt{2}}.$$

SOLUTION. Multiplying both numerator and denominator by $\sqrt{3}-\sqrt{2}$, the given fraction takes the form

$$\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}=\frac{\sqrt{3}-\sqrt{2}}{3-2}=\frac{\sqrt{3}-\sqrt{2}}{1}$$
$$=\sqrt{3}-\sqrt{2}. \quad \textit{Ans.}$$

EXAMPLE 3. Rationalize the denominator in the fraction

$$\frac{3\sqrt{5}+2\sqrt{2}}{\sqrt{5}-\sqrt{2}}.$$

SOLUTION. Multiplying both numerator and denominator by $\sqrt{5}+\sqrt{2}$, we obtain

$$\frac{(\sqrt{5}+\sqrt{2})(3\sqrt{5}+2\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2}=\frac{3\cdot 5+3\sqrt{10}+2\sqrt{10}+2\cdot 2}{5-2}=\frac{19+5\sqrt{10}}{3}. \quad \textit{Ans.}$$

A careful examination of the examples above shows that if the denominator is of the form $\sqrt{a}$; that is, consists of a single surd, the fraction may be rationalized by simply multiplying both numerator and denominator by $\sqrt{a}$. If, on the other hand, the denominator has either of the binomial forms $\sqrt{a}+\sqrt{b}$, or $\sqrt{a}-\sqrt{b}$, the fraction should have its numerator and denominator multiplied by $\sqrt{a}-\sqrt{b}$ in the first case and by $\sqrt{a}+\sqrt{b}$ in the second.

EXERCISES

Write each of the following expressions with rationalized denominators.

1. $\dfrac{\sqrt{3}}{\sqrt{6}}$.
2. $\dfrac{\sqrt{7}}{\sqrt{10}}$.
3. $\dfrac{\sqrt{a}}{\sqrt{b}}$.
4. $\dfrac{\sqrt{3\,bx}}{\sqrt{2\,ax}}$.
5. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
6. $\dfrac{1}{\sqrt{3}-1}$.

7. $\dfrac{2}{\sqrt{6}-\sqrt{3}}$.

8. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

9. $\dfrac{3\sqrt{3}-2\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$.

10. $\dfrac{2+\sqrt{5}}{2\sqrt{7}}$.

11. $\dfrac{3\sqrt{a}-4\sqrt{b}}{2\sqrt{a}-3\sqrt{b}}$.

12. $\dfrac{\sqrt{x+1}+3}{\sqrt{x+1}+2}$.

* **149. Finding the Value of Fractions Containing Surds.** Suppose we wish to find the value of the expression

$$\frac{1}{\sqrt{3}+\sqrt{2}}.$$

We begin by rationalizing the denominator, thus making the fraction take the form $(\sqrt{3}-\sqrt{2})/1$, or $\sqrt{3}-\sqrt{2}$. All we now need to do is to look up in the tables the values of $\sqrt{3}$ and $\sqrt{2}$ and subtract the latter from the former. The entire work may be written in the form

$$\frac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{3}-\sqrt{2}=1.73205^{+}-1.41421^{+}=0.31784^{+}. \quad \textit{Ans.}$$

If we had not first rationized the denominator, we should have had to find the value of

$$\frac{1}{1.73205^{+}+1.41421^{+}}, \quad \text{or} \quad \frac{1}{3.14626^{+}},$$

which would thus compel us to *divide* 1 by 3.14626^{+}. Note how much more difficult this would be than the above, where all we needed to do in the end was to *subtract* 1.41421 from 1.73205, which is very simple.

This illustrates the general fact that *in finding the value of a fraction, its denominator* (if it contains surds) *should first be rationalized.*

* EXERCISES

Find (by rationalizing denominators and then using the tables) the approximate value of the following expressions.

1. $\dfrac{3}{\sqrt{2}}$.

2. $\dfrac{2\sqrt{5}}{3\sqrt{2}}$.

3. $\dfrac{1}{\sqrt{2}-1}$.

4. $\dfrac{1+\sqrt{2}}{2-\sqrt{3}}$.

5. $\dfrac{1}{\sqrt{300}}$.

6. $\dfrac{3\sqrt{3}-4}{4\sqrt{3}-5}$

CHAPTER XVI

QUADRATIC EQUATIONS

PART I. PURE QUADRATICS

150. Quadratic Equation. An equation which contains the unknown letter to the second (but no higher) power is called a ***quadratic equation***, or briefly, a ***quadratic***.

Thus, $3x^2-2x=4$ and $x^2+2x+1=0$ are quadratics, but $3x-1=0$ and $2x^3-4x^2+x=6$ are not.

151. Pure and Affected Quadratics. When the quadratic contains the second power only of the unknown letter, it is called a ***pure quadratic***.

Thus, $x^2=25$, $6x^2-24=0$ and $ax^2=bc$ are pure quadratics, but $x^2-x=25$ is not.

When the quadratic equation contains both the first and second powers of the unknown letter, it is called an ***affected quadratic***, or a ***general quadratic***.

Thus, $x^2+2x-15=0$ is an affected quadratic, but $2x^2-27=0$ is not.

152. Solution of Pure Quadratics. The following two examples suffice to show how the solution of any pure quadratic may be obtained.

EXAMPLE 1. Solve the equation

$$x^2-25=0.$$

SOLUTION. $\qquad x^2-25=0.$

Transposing, we find

$$x^2=25.$$

Taking the square root of both members of the last equation gives

$$x = \pm 5. \quad \textit{Ans.}\dagger$$

The meaning of this answer is that x may have either the value $+5$ or -5. That is, the given equation has the *two* different roots 5 and -5. See § 137.

CHECK. Substituting 5 for x in the given equation gives $5^2=25$, which is correct. Similarly, if we substitute -5 for x we have $(-5)^2=25$, which also is true.

EXAMPLE 2. Solve the equation

$$2x^2-30=0.$$

SOLUTION. Transposing and dividing by 2 gives

$$x^2=15.$$

Taking the square root of both members gives

$$x = \pm\sqrt{15}. \quad \textit{Ans.}$$

It should be carefully noted that these answers are different in character from those in Example 1 because we are here asked to extract the square root of 15, and this can be done only *approximately*. (See § 139.) Since the tables give $\sqrt{15}=3.87298$, our answers, correct to five decimal places, are $x=\pm 3.87298$.

CHECK. $2(\sqrt{15})^2-30=2\times 15-30=30-30=0$, as required.
$2(-\sqrt{15})^2-30=2\times 15-30=30-30=0$, as required.

An examination of the examples above shows that we have the following rule.

RULE FOR SOLVING A PURE QUADRATIC. *Solve for x^2, then take the square root of the result.*

There will always be **two** *roots, the one being the negative of the other.*

† Strictly speaking, when we extract the square root of both members of the equation $x^2=25$ we get $\pm x = \pm 5$ instead of $x=\pm 5$. But to say that $-x=\pm 5$ is the same as to say that $x=\pm 5$, so it suffices to write simply $x=\pm 5$ to cover all possible cases.

EXERCISES

Solve and check your answers in the following. If you meet with a surd, find its approximate value by use of the tables.

1. $x^2-121=0.$

2. $6x^2-72=0.$

3. $12x^2-64=4x^2+8.$

4. $2x^2-5=0.$

5. $22x^2-352=0.$

6. $x^2-3-\frac{x^2-4}{4}=\frac{5x^2-10}{7}.$

7. $(11x)^2=513+(8x)^2.$

8. $\frac{x^2}{5}=\frac{x^2}{4}-20.$

9. $2(x-5)+3x(x-1)=17-x.$

10. $\frac{2x+7}{x+6}=\frac{3x+15}{x+13}.$

11. $\frac{4x^2+18}{9}=x^2-3.$

12. $\frac{10}{1-2x}+\frac{10}{1+2x}=36.$

13. $ax^2-c=1.$ $x=\pm\sqrt{\frac{1+c}{a}}.$ *Ans.*

14. $x^2-a=0.$

15. $2x^2-a(x+b)=x(x-a).$

16. $\frac{1}{a-x}+\frac{1}{a+x}=\frac{b+x^2}{a^2-x^2}.$

17. $\frac{2a+x}{2a-x}+\frac{2a-x}{2a+x}=b.$

18. $\sqrt{2x^2+x}=x+\frac{1}{2}.$

[HINT. See § 138.]

19. $\sqrt{2(x^2+ax)}=x+a.$

EXERCISES — APPLIED PROBLEMS

If surds occur in the answers, get their approximate values by means of the tables.

1. If three times the square of a certain number be diminished by 52, the result is 20 more than the square of the number itself. What is the number?

2. One side of a certain right triangle is 9 inches long, and the hypotenuse is 15 inches long. How long is the other side?

[HINT. Work by algebra, using the principle in Ex. 20, p. 21.]

3. How far apart are the two diagonally opposite corners of a room that is 12 feet wide and 15 feet long?

4. What is the length of the longest umbrella that can be placed in the bottom of a trunk the inside of which is 32 inches by 20 inches?

5. A square field contains $5\frac{5}{6}$ acres. How many rods of fence will be required to inclose it?

6. The sides of a 90-acre field are to each other as $2\frac{1}{2}$ is to 4. Find the length and breadth.

[HINT. Let x = the length and y = the width. Then we must have $x/y = (2\frac{1}{2})/4$. Solving this for y gives $y = \frac{8}{5}x$. The area of the field, or xy, thus becomes $\frac{8}{5}x^2$. Now find x.]

7. Find the mean proportional between 27 and 12.

[HINT. Let x be the number desired. Then we must have $27/x = x/12$. See § 116.]

8. Find the mean proportional between 16 and 21. State any essential difference between your answer and that obtained for Ex. 7.

9. In the semicircle ABC the line PQ is perpendicular to the base and divides it into parts which are 5 inches long and 8 inches long respectively. How long is PQ?

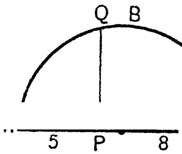


FIG. 78.

[HINT. See Ex. 8, p. 197.]

10. From a certain point P outside a circle a secant is drawn cutting the circle at the points Q and R. If PQ is 6 inches and PR is 11 inches how long is the tangent PT?

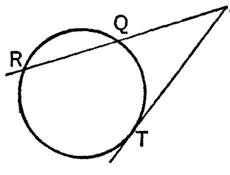


FIG. 79.

[HINT. See Ex. 10, p. 197.]

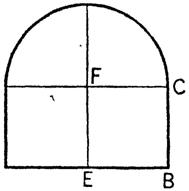


FIG. 80.

11. If you have a board 20 inches square how wide a strip must be taken from all sides so that the remaining square shall have one half the area of the original piece? $5(2-\sqrt{2})$ in. *Ans.*

12. We know that a window whose lower part is a rectangle and whose upper part is a semicircle admits the most light when the width and height are the same. If the area of such a window is $32\frac{1}{7}$ square feet, what is the width?

[HINT. Take $\pi = 3\frac{1}{7}$.]

13. An 8-inch square was taken from each corner of a square piece of tin. The sides were then turned up so as to form a box containing 1152 cubic inches. How long was the piece of tin?

14. What is the formula for the length of the tangent from a point to a circle if the secant from that point is divided by the circle into segments (parts) equal respectively to a and b?

15. What is the formula for the radius of the circle whose area is A? Apply your answer to find the radius in case the area is $37\frac{1}{2}$ square inches?

16. One of the sides of a certain triangle is a units long. What is the formula for the corresponding side of a triangle which is similar to the first one, but whose area is b times as great?

[HINT. See § 115.]

17. Find the formula for the radius of the sphere whose surface is n times as great as that of the sphere whose radius is r.

PART II. SOLUTION OF AFFECTED QUADRATICS

153. Solution of Affected Quadratics by Factoring. An affected quadratic may often be solved by factoring. This was explained in § 68, which the pupil should now read again, but the following examples will further illustrate the method.

EXERCISES

Solve by factoring and check your answers for Exs. 1–10.

1. $x^2+8x-48=0$.

SOLUTION. $x^2+8x-48=(x-4)(x+12)$. (See § 58.)

The desired solutions are therefore those obtained by solving the simple equations $x-4=0$ and $x+12=0$. (See § 68.)

Hence, the solutions are $x=4$ and $x=-12$. *Ans.*

CHECK. Substituting 4 for x in the given equation gives $4^2+8\times 4-48$, which reduces to $16+32-48$, and this $=0$ as required. Similarly, if we substitute -12 for x we obtain $(-12)^2+8\times(-12)-48$ which $=144-96-48=0$, as required.

2. $x^2-x-12=0$.

3. $x^2-12x+35=0$.

4. $x^2-7x=8$.

[HINT. Remember to transpose the 8 so as to have the right-hand member zero. See § 68.]

5. $3-x^2=-2x$.

6. $6-x^2=x$.

7. $18x^2+18=-36x$.

8. $6x^2-12x=0$.

9. $\dfrac{x^2}{3}+8=-\dfrac{11}{3}x$.

10. $x-\dfrac{5}{x+4}=0$.

11. $\dfrac{2}{x^2+2x+1}=\dfrac{1}{2}$.

12. $\dfrac{x^2}{x-2}=\dfrac{-4}{x-2}-5$.

13. $\dfrac{x-2}{x+2}+\dfrac{x+3}{x-3}=-5$.

14. $\dfrac{16}{x-6}=x$.

15. $(x+4)(x-2)=11(x-2)$.

[HINT. Write as $(x-2)[(x+4)-11]=0$ and apply the principle in § 68.]

16. $3(x+1)(x-4)+4(x-4)=0$.

17. $(x+2)^2+3(x+2)+2=0$.

[HINT. Solve first for $(x+2)$.]

18. $(x-1)^2-6(x-1)=-8$.

19. $\frac{1}{x^2}-\frac{9}{x}+20=0$.

[HINT. Solve first for $1/x$.]

20. $(x+a)(x+b)+4(x+a)=0$.

154. Solution by Completing the Square. We often meet with quadratics, such as

$$x^2+7x-5=0,$$

which we cannot solve by factoring, as in § 153. The difficulty here is that we cannot factor x^2+7x-5. However, this quadratic and *all* others (whether they be solvable by factoring or not) can be solved by a process called ***completing the square.*** How this is done will be best understood from a careful study of the following examples.

NOTE. At this point the pupil should review §§ 58–63.

EXAMPLE 1. Solve the equation

$$x^2+6x=16.$$

SOLUTION. The first member of this equation, or x^2+6x, would become a perfect trinomial square if 9 were added to it (see § 63). Our first step, therefore, is to add 9 to *both* sides of the equation, thus "completing the square" on the left side and at the same time not destroying the equality of the two members. This gives the equation

$$x^2+6x+9=25,$$

or,

$$(x+3)^2=25.$$

Taking the square root of both members of the last equation gives

$$x+3=\pm 5.$$

Therefore, we must either have $x+3=5$ or $x+3=-5$.

Solving the last two equations separately gives as the desired solutions $x=2$ and $x=-8$. *Ans.*

CHECK. Substituting 2 for x in the first member of the given equation gives $2^2+6\times 2$ or $4+12$ which is 16 as required. Similarly, when $x=-8$ the first member becomes $(-8)^2+6\times(-8)$, or $64-48$, which is 16 as required.

EXAMPLE 2. Solve $x^2-8\,x+14=0$.

SOLUTION. Transposing, we find

$$x^2-8\,x=-14.$$

Completing the square by adding 16 to both sides gives

$$x^2-8\,x+16=2, \text{ or } (x-4)^2=2.$$

Taking the square root of both members,

$$x-4=\pm\sqrt{2}.$$

Solving the last two equations gives

$$x=4+\sqrt{2} \text{ and } x=4-\sqrt{2}. \quad \textit{Ans.}$$

CHECK. With $x=4+\sqrt{2}$, the first member of the given equation becomes $(4+\sqrt{2})^2-8(4+\sqrt{2})+14$. By Formula V, p. 101, this may be written

$$(16+8\sqrt{2}+2)-8(4+\sqrt{2})+14, \text{ or } 16+8\sqrt{2}+2-32-8\sqrt{2}+14.$$

Here the $8\sqrt{2}$ and $-8\sqrt{2}$ cancel, while the rest of the expression (namely, $16+2-32+14$) reduces to 0. Whence, if $x=4+\sqrt{2}$ the first member reduces to 0, as required.

Likewise, when x has its other value, namely $4-\sqrt{2}$, the first member of the given equation may be shown at once to reduce to 0 as required.

NOTE. Since the solutions obtained above for Example 2, namely $x=4+\sqrt{2}$ and $x=4-\sqrt{2}$, contain the *surd* $\sqrt{2}$, they cannot be expressed exactly (see § 139), but we can express them *approximately*. Thus, the Tables give $\sqrt{2}=1.41421$, so that the two solutions are approximately $4+1.41421$ and $4-1.41421$, which reduce to 5.41421 and 2.58579. *Ans.*

EXAMPLE 3. Solve $3x^2+8x=15$.

SOLUTION. Dividing through by 3, we find

$$x^2+\tfrac{8}{3}x=5.$$

Completing the square by adding $(\tfrac{4}{3})^2$, or $(\tfrac{16}{9})$ to both members gives

$$x^2+\tfrac{8}{3}x+(\tfrac{4}{3})^2=5+\tfrac{16}{9}=\tfrac{61}{9}$$

or,

$$(x+\tfrac{4}{3})^2=\tfrac{61}{9}.$$

Taking the square root of both members gives

$$x+\tfrac{4}{3}=\pm\sqrt{\tfrac{61}{9}}=\pm\tfrac{1}{3}\sqrt{61}.$$

Therefore the two solutions are

$$x=-\tfrac{4}{3}+\tfrac{1}{3}\sqrt{61}, \text{ and } x=-\tfrac{4}{3}-\tfrac{1}{3}\sqrt{61}. \quad \textit{Ans.}$$

NOTE. These two answers may be written together in the condensed form $\frac{1}{3}(-4\pm\sqrt{61})$. By looking up the value of $\sqrt{61}$ in the Tables, the value of each answer may be determined approximately, as explained in the Note to Example 2.

155. Summary of Results and Rule. It is now to be carefully observed that in each of the three examples worked in § 154 the first step consisted in reducing the given equation to the type form

$$x^2+px=q,$$

where p and q are certain known numbers.

Thus, in Example 3 we first put the equation in the form $x^2+\frac{8}{3}x=5$. Here $p=\frac{8}{3}$ and $q=5$.

The next step was to complete the square by adding to both members the square of *half* the coefficient of x; that is, we added $(p/2)^2$ to both members.

Thus, in Example 3, where $p=8/3$, we have $p/2=4/3$; hence the number added to both sides to complete the square was $(4/3)^2$.

After this, the equation was always such that we could extract the square root of the left member. When we did so and equated (placed equal to each other) the results, we had

two simple equations from which the two solutions could be found at once.

This may now be summarized into the following rule.

Rule for Solving any Quadratic by Completing the Square.

1. *Reduce the equation to the form*

$$x^2+px=q.$$

2. *Complete the square by adding* $(p/2)^2$ *to each member.*

3. *Extract the square root of both members of the equation thus obtained and equate the results. This yields two simple equations from which the two desired values of* x *may be found.*

EXERCISES

Solve the following equations, and check your answers in the first five.

1. $x^2+4x=12$.
2. $x^2-6x=16$.
3. $x^2-20x=21$.
4. $x^2-18x=-80$.
5. $x^2+3x=\frac{7}{4}$.
6. $x^2-7x=-10$.
7. $x=1-x^2$.
8. $x^2-\frac{8}{3}x=1$.
9. $3x^2-4x=1$.
10. $5x^2+25x=-9$.
11. $6x^2=3x+45$.
12. $x^2-4x=6(x+4)$.
13. $2x(x+4)=42$.
14. $(3x-2)^2=6x+11$.
15. $x+\dfrac{15}{x}=16$.
16. $\dfrac{9x^2}{4}-3x=-1$.
17. $\dfrac{x^2-24}{x}=10$.
18. $\dfrac{72}{x-6}=x$.
19. $\dfrac{x-3}{x+5}=\dfrac{6-x}{x-2}$.
20. $-2a^2=ax-x^2$.
    $x=2a,\ x=-a$. *Ans.*
21. $12x^2-3ax=3a^2$.

For further exercises on this topic, see the review exercises, *p.* 285, *and Appendix, p.* 310.

156. Quadratics of Special Form. Suppose we wish to solve the quadratic $4x^2+4x=15$. Here the coefficient of x^2 is 4, or 2^2, and is therefore a perfect square. In such a case we do not need to first divide the equation through by 4, as the Rule of § 155 would require, but we can complete the square *immediately* in the equation as it stands. Thus, by adding 1 to both members we have

$$4x^2+4x+1=16, \text{ which is the same as } (2x+1)^2=16.$$

Extracting the square root of both members of the last equation gives $2x+1=\pm 4$, so the two solutions are those of the simple equations $2x+1=4$ and $2x+1=-4$; that is, they are $x=3/2$ and $x=-5/2$. *Ans.*

All quadratics such as the one just mentioned are of the type form

$$n^2x^2+bx=k,$$

where n, b, and k are certain determinate numbers, and *we can immediately complete the square by adding* $(b/2n)^2$ *to both members.*

Thus, in the quadratic $25x^2-30x=72$ we have $n=5$ (because $n^2=25$) and we have $b=-30$, $k=72$. So the square can be immediately completed by adding $(-30/10)^2=(-3)^2=9$ to both members. This gives $25x^2-30x+9=81$, or $(5x-3)^2=81$, from which we find the solutions $x=\frac{12}{5}$ and $x=-\frac{6}{5}$. This amounts to the same thing as the process of § 155, as can be seen in this example by first dividing both sides of the given equation by 25.

It should be observed that if $n=1$ the type form we have just been considering reduces to the one employed in § 155.

EXERCISES

Solve the following by the method of completing the square explained in § 156. Check your answers in the first five.

1. $4x^2+4x=8$. **2.** $4x^2-4x=15$. **3.** $9x^2-12x=5$.

4. $16x^2+8x-1=0$. $\frac{1}{4}(-1\pm\sqrt{2})$. *Ans.*

5. $16x^2-10x=-\frac{21}{16}$.

6. $36x^2+6x=\frac{3}{4}$.

7. $2x^2+6x=\frac{7}{2}$.

[HINT. First multiply both members by 2 so as to get a perfect square for the coefficient of x^2.]

8. $3x^2-2x=5$.

[HINT. Multiply both members by 3.]

9. $5x^2-20x+14=0$.

10. $8x^2-5x-1=2$.

***157. Solution by the Hindu Method.** A simple way, preferred by many, for completing the square in any quadratic is the one called the "Hindu Method." It consists of two steps:

1. Multiply both members by four times the coefficient of x^2.
2. Add to both members of the new equation the square of the original coefficient of x.

EXAMPLE. Solve $3x^2-2x=1$.

Multiplying through by four times the coefficient of x^2, that is by 12, gives

$$36x^2-24x=12.$$

Adding the square of the original coefficient of x to both sides, that is, adding $(-2)^2$ or 4 to both sides, gives

$$36x^2-24x+4=16.$$

The first member is now a perfect square, being equal to $(6x-2)^2$. Therefore, upon extracting square roots, we obtain

$$6x-2=4, \text{ and } 6x-2=-4.$$

Solving the last two equations, gives $x=1$, and $x=-\frac{1}{3}$. *Ans.*

EXERCISES

Solve each of the following equations by any method.

1. $x^2+10x+21=0$.
2. $x^2-5x=24$.
3. $2x^2+7x=60$.
4. $x^2-4x=117$.
5. $8x=x^2-180$.
6. $5x^2-3x-2=0$.

7. $8x^2-10x=3$.

8. $2x^2-11x+12=0$.

9. $2x^2+3x=27$.

10. $4x^2-3x-3=0$.

11. $3x^2+x-200=0$.

12. $2x^2+5x+2=0$.

13. $1-3x=2x^2$.

14. $4=x(3x+2)$.

15. $x+\frac{1}{x}-\frac{5}{2}=0$.

16. $\frac{x}{9(x-1)}=\frac{x-2}{6}$.

17. $\frac{x^2}{4}-\frac{2x}{3}=28$.

18. $\frac{x}{x+2}+\frac{1}{2}=\frac{x+2}{2x}$.

19. $x^2-ax=bx-ab$. $x=a$ or $x=b$. *Ans.*

20. $x^2+ax=ac+cx$.

21. $x^2=4ax-2a^2$. $a(2\pm\sqrt{2})$. *Ans.*

22. $\frac{x}{x-1}-\frac{x}{x+1}=m$. $\frac{1}{m}(1\pm\sqrt{1+m^2})$. *Ans.*

23. $\frac{1}{ax+4}=1-\frac{ax-4}{16}$.

EXERCISES — APPLIED PROBLEMS

1. The square of a certain number is 24 more than twice the number. Find the number.

[Hint. Remember that there should be two solutions.]

2. The square of a certain number is $\frac{1}{9}$ less than $\frac{2}{3}$ of the number. Find the number.

3. The difference between the cube of a certain number and three times the square of the number is equal to four times the number itself. Find the number.

[Hint. After forming the equation, divide both sides by x.]

4. The sum of the squares of two consecutive numbers is 1013. What are the numbers?

[Hint. For the definition of consecutive numbers, see Ex. 36, p. 62.]

5. The product of two consecutive numbers is 272. What are the numbers?

6. The hypotenuse of a right triangle is 10 inches long and the sum of its sides is 14 inches. Find the lengths of the sides.

7. One side of a right triangle is 4 in. longer than the other. If the hypotenuse is 20 in. long, how long are the sides?

[HINT. If x represent the shorter side, the equation here becomes $x^2+(x-4)^2=400$ and in solving this one of the solutions turns out to be *negative*. But a negative number can have no meaning in such an example as this, so we keep only the positive solution. This frequently happens in applied problems containing quadratics, so the pupil must always be on his guard to keep only such solutions as can actually fit a given example.]

8. An ordinary gable roof has the form (cross section) indicated in the figure. Suppose the "run" is 8 feet and the "rafter" 10 feet. How much greater would the "rise" be if an 11-foot rafter had been used?
$\sqrt{57}-6$, or 1.54983^+ ft. *Ans.*

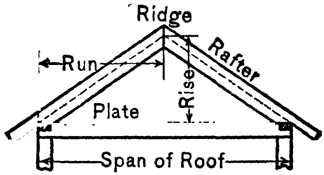


FIG. 81.

9. A gardener spades a bed 40 feet long and 20 feet wide. He then decides to make the bed three times as large by adding to all sides a strip of the same width throughout. How wide must the strip be?

10. A rectangular plot of ground measures 160 feet by 40 feet. By how many feet must the length and breadth be equally increased so that the area becomes increased by 10,000 square feet?　　$100(\sqrt{2}-1)$, or 41.421^+ ft. *Ans.*

11. A coach wishes to increase the length and breadth of a certain athletic field by the same amount in such a way that the diagonal line across the field will become increased by 50 feet. The field is now 400 feet by 300 feet. How many feet must be added to each dimension?　　35.68^+ ft. *Ans.*

12. A circular swimming pool is surrounded by a walk 6 feet wide. The walk contains half as much area as the pool. Find (approximately) the radius of the pool.

13. If a train had its speed increased by 5 miles an hour, it could shorten its time for running 180 miles by 30 minutes. What is the rate in miles per hour?

14. The switchboard in a telephone office is an arrangement by which any one person who has a telephone may be connected with any other person in the system. If the number of persons in the system is n, it is known that the total number N of connections possible on the switchboard is given by the formula $N = n(n-1)/2$. If 53,628 connections are possible, how many telephones are there in the system?

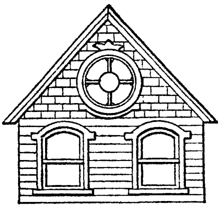


FIG. 82.

15. Figure 82 represents the gable end of a house with a circular window in it. The two points on the under side of the roof that are nearest the window are each $1\frac{1}{2}$ ft. from the outer circumference of the window frame; and they are 6 ft. from the under edge of the peak. If the diameter of the window frame is 6 ft., how far is its center beneath the under edge of the peak?

16. Figure 83 represents a pattern frequently used in window designs, consisting of a square $ABCD$ with a semicircle EFG mounted upon it, the diameter GE of the semicircle being slightly less than one of the sides of the square. If the shoulders AG and DE are to be each 1 foot long and the total lighting surface is to be 88 square feet, find how long each side of the square must be made. $8\frac{6}{13}$ ft. *Ans.*

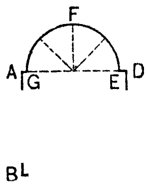


FIG. 83.

17. Solve Ex. 16 when the lower part of the window, instead of being a square, is to be a rectangle 3 feet higher than wide (other conditions remaining the same).

18. As the radius of a certain sphere was lengthened out 2 feet, the surface of the sphere became exactly double its original value. What was the original radius?

$2(1+\sqrt{2})$ ft. *Ans.*

[HINT. The area of the sphere whose radius is r is $4\pi r^2$.]

19. The figure represents a familiar form of pendant, or watch charm, consisting of a circular (or sometimes spherical) disk supported by two equal metal strips soldered to the circumference and meeting in a point above. Placing $PT=t$ and calling the diameter of the circle d, show that the formula for the length of PS is

$$PS=\tfrac{1}{2}(-d+\sqrt{d^2+4t^2}).$$

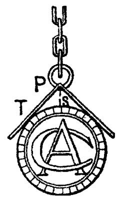


FIG. 84.

Observe that by means of this formula the small length PS (which it is usually difficult to measure accurately) can be determined by measuring the larger and more accessible distances t and d. Explain how.

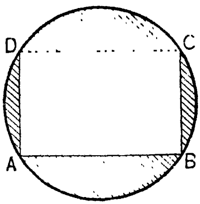


FIG. 85.

20. The figure shows a rectangle whose base is 2 feet longer than its height. The rectangle is surrounded (circumscribed) by a circle. What must the dimensions of the rectangle be if the shaded area is to be 20 square feet?

[HINT. The diagonal of the rectangle is a diameter of the circle.]

$$1+\sqrt{\frac{38-\pi}{\pi-2}}\text{ ft. and } -1+\sqrt{\frac{38-\pi}{\pi-2}}\text{ ft.}\quad \textit{Ans.}$$

PART III. SUPPLEMENTARY TOPICS †

158. The Graphical Solution of Quadratic Equations. Suppose we have the quadratic equation $x^2-3x-4=0$. Let us represent the left member by the letter y; that is, let us write

$$y=x^2-3x-4.$$

Now, if x is given some value, this equation determines a corresponding value for y. For example, if $x=0$, then $y=0^2-3\times 0-4=-4$. Again, if $x=1$, $y=1^2-3\times 1-4=-6$. The table below shows a number of x values with their corresponding y values determined in this way:

x	0	1	2	3	4	5	−1	−2
y	−4	−6	−6	−4	0	6	0	6

This table is at once seen to resemble those in §§ 122, 124. Like them, it has a certain graph to correspond to it. This graph is obtained by first drawing axes XX and YY and then plotting (in the sense explained in § 122) each of the points x, y which the table contains and finally drawing the smooth curve which passes through all such points. The curve thus obtained is the *graph* of the given quadratic $x^2-3x-4=0$.

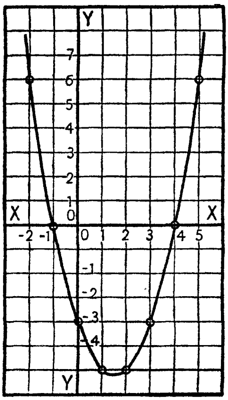


FIG. 86.

It is to be observed that the graph as thus determined is not a straight line and is therefore much different in character from the graph of a linear equation (compare § 123). And it is

† This part contains a brief introduction to several topics which are considered in further detail in Vol. II.

especially important to notice that it cuts the XX axis in *two* points whose x-values are $x=-1$ and $x=4$ respectively. The two values of x determined in this graphical way are really the two solutions of the given quadratic equation, $x^2-3x-4=0$, for they are those values of x that make $y=0$; that is, which make $x^2-3x-4=0$.

In view of what we have just seen regarding the graph of the special quadratic $x^2-3x-4=0$, we may now state the following general facts:

To obtain the graph of a quadratic equation, first place y equal to the left member of the equation (it being understood that the right member is 0). Then assign various values to x and determine the corresponding values of y. Plot the points (x, y) thus obtained and draw a smooth curve through them, as in § 118.

The x-values of the two points where the graph cuts the XX axis will be the roots of the given quadratic.

EXERCISES

Draw the graph of each of the following quadratics, noting especially where it cuts the XX axis. State any conclusions you obtain in this way regarding the *solutions* of the given equation and check the correctness of your statements by actually finding the solutions by one of the methods explained in §§ 153–157.

1. $x^2-7x+6=0$.

2. $x^2-2x-3=0$.

3. $x^2-5x=-6$.

[HINT. Remember to write as $x^2-5x+6=0$.]

4. $x^2+5x=-6$.

5. $2x^2+3x-9=0$.

* **159. Quadratics Having Imaginary Solutions.** Suppose we have the quadratic $x^2=-1$. This is a pure quadratic (§ 150) and hence we can immediately solve it by taking the square root of both members, which gives as the desired solutions $x=\pm\sqrt{-1}$. But $\sqrt{-1}$ means the number whose square is -1 and *there is no such number among all those (positive or negative) which we have thus far met in algebra.* In fact, the square of any number, positive or negative, we know to be *positive* (see § 26). Therefore, in any such case as this we say that the solutions of the given quadratic are *imaginary* and we speak of the numbers themselves which, like $\sqrt{-1}$, enter into algebra in this way as *imaginary numbers.* Such numbers are "imaginary" only in the sense that we have not met with them before.

As an example of an affected quadratic having imaginary roots we may consider the equation $x^2-6x+15=0$. If we proceed to solve this by completing the square, as in § 154, the work is as follows:

Transposing,

$$x^2-6x=-15.$$

Adding 9 to both sides to complete the square gives

$$x^2-6x+9=-6 \text{ or } (x-3)^2=-6.$$

Extracting the square root of both sides,

$$x-3=\pm\sqrt{-6}.$$

Therefore

$$x=3+\sqrt{-6}, \text{ or } x=3-\sqrt{-6}. \quad \textit{Ans.}$$

Both of these roots are seen to be imaginary because they contain the expression $\sqrt{-6}$.

All numbers such as we have been dealing with in the Chapters before this (including the surds and radicals) are called *real* numbers in distinction to the imaginary numbers just mentioned.

EXERCISES

Find (by solving) whether the solutions of each of the following quadratics are real or imaginary.

1. $x^2+4=0$.

2. $2x^2+6=10$.

3. $2x^2+11=10$.

4. $3x^2+2=0$.

5. $x^2+4x+2=0$.

6. $x^2+5x=6$.

7. $3x^2-x+1=0$.

8. $6x^2+4x+3=0$.

*** 160. Determining Graphically whether Solutions are Real or Imaginary.** If we take the equation $x^2-6\,x=-15$, which was shown in § 159 to have imaginary roots, and ask what the graph of such an equation resembles, there is no difficulty in answering. We proceed, as stated in § 158, to put $y=x^2-6\,x+15$ and form a table of x values with their corresponding y values such as given below:

x	-2	-1	0	1	2	3	4	5	6
y	31	22	15	10	7	6	7	10	15

Plotting the points x, y of the table in the usual way and drawing the curve through them gives a graph such as shown in the diagram at the right. Thus the graph is in no essential different from those met with before in § 158 *except that it does not cut the XX axis at all* (instead of cutting it in two points, as was always the case when the roots of the given quadratic were real).

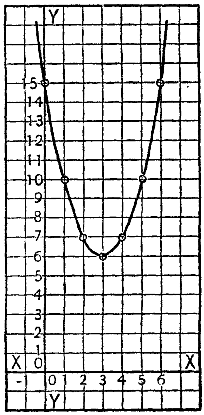


FIG. 87.

What we have thus seen for the special quadratic $x^2-6\,x+15=0$ is true in general; that is, *whenever the roots of any quadratic are imaginary, the graph does not cut the XX axis at all.*

It follows that one way to tell whether the solutions of a given quadratic are real or imaginary is to simply draw the graph and see whether or not it cuts the XX axis.

EXERCISES

Find by drawing the graph whether the roots of the following quadratics are real or imaginary.

1. $x^2-6\,x+12=0$.	**5.** $x^2-2\,x-8=0$.
2. $x^2+6\,x+12=0$.	**6.** $x^2-2\,x+6=0$.
3. $x^2+4\,x-5=0$.	**7.** $3\,x^2+4\,x+1=0$.
4. $x^2+4\,x+9=0$.	**8.** $3\,x^2+4\,x+2=0$.

* **161. Solution of Quadratics by Formula.** Every quadratic may be regarded as an equation of the form

(1) $$ax^2+bx+c=0,$$

where a, b, and c are known numbers.

For example, in $2\,x^2+3\,x+5=0$ we have $a=2$, $b=3$, $c=5$; **again**, in $x^2-\frac{1}{2}\,x=3$ we have $x^2-\frac{1}{2}\,x-3=0$ so that $a=1$, $b=-\frac{1}{2}$, $c=-3$.

Since (1) thus represents *all* quadratics, it follows that if we can solve it as it stands (that is, regarded as a literal equation containing the known letters a, b, and c) we shall thereby arrive at general formulas for the solutions of any quadratic whatever. This can be done as follows:

Transposing c,

$$ax^2+bx=-c.$$

Multiplying through by $4\,a$,

$$4\,a^2x^2+4\,abx=-4\,ac.$$

Adding b^2 to both members,

$$4\,a^2x^2+4\,abx+b^2=b^2-4\,ac, \text{ or } (2\,ax+b)^2=b^2-4\,ac.$$

Extracting square roots,

$$2\,ax+b=\pm\sqrt{b^2-4\,ac}.$$

Whence (solving for x) the two solutions, which we will now call x_1 and x_2, are seen to be

$$\boldsymbol{x_1=\frac{-b+\sqrt{b^2-4\,ac}}{2\,a}}, \textit{ and } \boldsymbol{x_2=\frac{-b-\sqrt{b^2-4\,ac}}{2\,a}}.$$

These, then, are the general formulas sought, and the way in which they may be used in solving any particular quadratic is illustrated in the following examples:

EXAMPLE 1. Solve the quadratic $2\,x^2-11\,x+12=0$.

SOLUTION. Here we have $a=2$, $b=-11$ and $c=12$. The two solutions, as obtained by merely substituting these values of a, b, c into the above formulas, are therefore

$$x_1=\frac{11+\sqrt{(-11)^2-4\cdot 2\cdot 12}}{2\cdot 2}, \text{ and } x_2=\frac{11-\sqrt{(-11)^2-4\cdot 2\cdot 12}}{2\cdot 2},$$

which reduce to $\frac{11+5}{4}$ and $\frac{11-5}{4}$, or 4 and $\frac{3}{2}$. *Ans.*

EXAMPLE 2. Solve $4x^2+3x-2=0$.

SOLUTION. Here $a=4$, $b=3$, $c=-2$. Substituting these values into the general formulas for the solutions gives in the present case

$$x_1=\frac{-3+\sqrt{9+32}}{2\cdot 4} \text{ and } x_2=\frac{-3-\sqrt{9+32}}{2\cdot 4}, \text{ or } \frac{-3+\sqrt{41}}{8} \text{ and } \frac{-3-\sqrt{41}}{8}.$$

Ans.

EXAMPLE 3. Solve $5x^2+4x+1=0$.

SOLUTION. Here $a=5$, $b=4$, $c=1$ so that the solutions become

$$\frac{-4\pm\sqrt{16-20}}{10}, \text{ or } \frac{-4\pm\sqrt{-4}}{10}. \quad \textit{Ans.}$$

It is to be observed that these solutions are imaginary (§ 159).

EXERCISES

Solve *by formula* the following equations.

1. $2x^2+5x+2=0$.
2. $6x^2-7x=-2$.
3. $x(2x+3)=-1$.
4. $2x^2+3x-1=0$.
5. $3x^2+2x=4$.
6. $3x^2-6x=-2$.
7. $x^2-6x=-10$.
8. $x^2+4(x+3)=0$.

***162. Simultaneous Quadratic Equations.** In Chapter XIII we saw how to solve two simultaneous equations containing the unknown letters x and y in case each of the given equations is *linear;* that is, in case no higher power of x or y than the first occurs in either equation. However, we often meet with simultaneous equations that are not linear. While there is no general rule for solving such equations, a great many are reducible in such ways that we can solve them if we can solve quadratics.

EXAMPLE 1. Solve the simultaneous equations

$$\begin{cases} 4x^2-xy=y, \\ x+y=3. \end{cases}$$

SOLUTION. Here the second equation is linear, so we can at once find y *in terms of* x; that is, we have $y=3-x$. Substituting this value of y in the first equation gives

$$4x^2-x(3-x)=3-x,$$

which reduces to

$$4x^2-3x+x^2=3-x,$$

or

$$5x^2-2x=3.$$

Solving this quadratic in the usual way, we find

$$x=1 \text{ or } x=-\tfrac{3}{5}.$$

Substituting these values in the second of the given equations gives

$$y=2 \text{ or } y=\tfrac{18}{5}.$$

The solutions of the equations are therefore $(x=1,\ y=2)$ and $(x=-\frac{3}{5},\ y=\frac{18}{5})$. *Ans.*

CHECK. $\begin{cases} 4\times 1^2-1\times 2=2, \\ 1+2=3. \end{cases}$ $\begin{cases} 4(-\frac{3}{5})^2-(-\frac{3}{5})(\frac{18}{5})=\frac{36}{25}+\frac{54}{25}=\frac{18}{5}, \\ -\frac{3}{5}+\frac{18}{5}=\frac{15}{5}=3. \end{cases}$

EXAMPLE 2. Solve the simultaneous equations

$$\begin{cases} 3x^2+y^2=13, \\ x^2-y^2=3. \end{cases}$$

SOLUTION. These equations may be first solved for x^2 and y^2. Thus, adding the equations gives $4x^2=16$, and therefore $x^2=4$. Whence, from the second equation we have $y^2=1$.

Since $x^2=4$ and $y^2=1$ it follows that $x=\pm 2$ and $y=\pm 1$.

Thus, taking into account all the possible combinations of signs, we get *four* solutions as follows:

$(x=2,\ y=1)$; $(x=-2,\ y=1)$; $(x=2,\ y=-1)$; $(x=-2,\ y=-1)$. *Ans.*

CHECK. Substitution of any one of these pairs of values in the given equations shows at once that the equations are then satisfied.

EXERCISES

Solve

1. $\begin{cases} x^2+y^2=25, \\ x-y=-1. \end{cases}$

2. $\begin{cases} 3x^2+y^2=43, \\ x+y=7. \end{cases}$

3. $\begin{cases} 4x^2-9y^2=36, \\ 4x^2+9y^2=36. \end{cases}$

4. $\begin{cases} x^2-4y^2=4, \\ x^2+y^2=4. \end{cases}$

5. $\begin{cases} x-y=2, \\ xy=-1. \end{cases}$

6. $\begin{cases} 10x-3xy+y=0, \\ x-y+2=0. \end{cases}$

7. $\begin{cases} x^2+xy=12, \\ x-y=2. \end{cases}$

8. $\begin{cases} xy(x-2y)=10, \\ xy=10. \end{cases}$

9. $\begin{cases} 3x(y+1)=12, \\ 3x=2y. \end{cases}$

10. $\begin{cases} 3xy-10x-y=0, \\ 2-y+x=0. \end{cases}$

EXERCISES — APPLIED PROBLEMS

1. The sum of two numbers is 3 and the sum of their squares is 5. What are the numbers?

2. A piece of wire 24 inches long is bent into the form of a right triangle whose hypotenuse is 10 inches. What are the lengths of its sides? Work by using two unknown letters, x and y.

3. It takes 52 rods of fence to inclose a rectangular garden containing 1 acre. How long and how wide is the garden?

4. The area of a right triangle is 150 square feet and its hypotenuse is 25 feet long. How long are the sides?

5. The area of a rectangular garden is 1200 square feet and the diagonal across it measures 50 feet. What are the length and breadth?

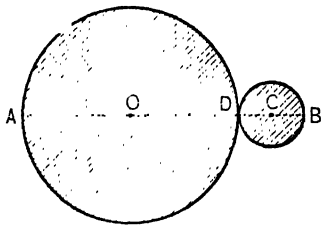


FIG. 88.

6. The mean proportional between two numbers is $\sqrt{21}$ and the sum of their squares is 58. Find the numbers.

7. Figure 88 shows two circles just touching (tangent to) each other, the smaller one being outside the larger one. If their combined areas (shaded in the figure) are 22 sq. ft. and the line AB which passes through the centers measures 6 feet, what is the radius of each circle? $\frac{1}{2}(3+\sqrt{5})$ ft. and $\frac{1}{2}(3-\sqrt{5})$ ft. *Ans.*

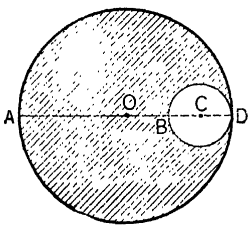


FIG. 89.

8. In Fig. 89, the inner circle is tangent to the outer one *internally*. If the shaded area is $47\frac{1}{2}$ sq. ft., and $AB=6$ ft., what is the radius of each circle?

9. Do *positive integers* exist differing by 3 and such that the sum of their squares is 117? If so, find them.

10. Answer Ex. 9 in case the sum of the squares is taken to be 120, other conditions remaining the same.

11. A rectangular swimming pool together with a platform around it 25 feet wide covered 37,500 square feet. The area of the platform was $\frac{7}{8}$ that of the pool. What were the dimensions of the pool?

12. Two men working together can complete a piece of work in $3\frac{8}{13}$ days. It would take one man 1 day longer than the other to do the work alone. In how many days can each man do the work alone?

[HINT. - See Ex. 24, p. 172.]

13. A sum of money on interest for one year at a certain rate brought \$7.50 interest. If the rate had been 1% less and the principal \$25 more, the interest would have been the same. Find the principal and rate.

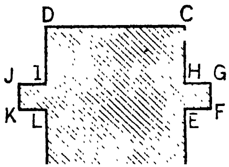


FIG. 90.

14. The figure shows a large square with two equal-sized smaller squares placed at opposite sides. If the shaded area is 72 square feet and the total perimeter of the figure is 40 feet, find the length of side of each square.

15. Solve Ex. 14 in case there are *four* small squares placed around the large square, one on each side, other conditions remaining the same as before.

16. If, in Fig. 91, the perimeter is 16 ft., and the area is 7 sq. ft., what is the length of side of each square? Compare Ex. 14.

17. A man traveled 30 miles. If his rate had been 5 miles an hour more, he could have made the journey in 1 hour less time. Find his time and rate.

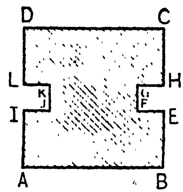


FIG. 91.

18. A and B each traveled 100 miles. A's speed was 5 miles an hour faster than B's and he arrived at the end of the journey 1 hour ahead of B. What was the rate of each?

Ans. A's rate = 25 mi. per hr.,
B's rate = 20 mi. per hr.

19. The figure shows a semicircle resting upon its base AB. At a certain point P on AB the perpendicular PG measures 4 inches, while at the point Q, which is 1 inch from P, the perpendicular QF measures 3 inches. Find the length of the base AB.

[HINT. Let $x = AP$, $y = PB$. Find x and y, then take their sum. See Ex. 8, p. 197.]

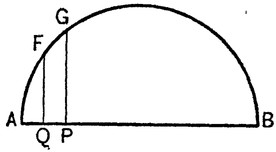


FIG. 92.

20. Show that the formulas for the length and width of the rectangle whose perimeter is a and whose area is b are

$$\tfrac{1}{4}(a+\sqrt{a^2-16b}) \text{ and } \tfrac{1}{4}(a-\sqrt{a^2-16b}).$$

EXERCISES — REVIEW OF CHAPTERS XIV–XVI

1. What is meant by the square root of a number?

2. How many square roots does a number have? Illustrate your answer by stating the square roots of 64; of 13.

3. State (by means of the tables) the approximate values of $\sqrt{30}$, $\sqrt{72}$, $\sqrt{84}$, $\sqrt{136}$.

4. How many terms has the square of a binomial?

5. Supply a third term to a^2+2ab that shall make the resulting trinomial a perfect square; to a^2+b^2; to $2ab+b^2$.

6. Complete the square in each of the following expressions:

(a) x^2-12x. (c) x^2+7x. (e) x^8-6x^4. (g) $25x^6-40x^3$.
(b) x^2+8x. (d) x^2-3x. (f) $9x^2+6x$. (h) $16x^4-16x^2$.

7. Find the square root of $9x^2-30ax-3a^2x+25a^2+5a^3+\frac{1}{4}a^4$.

8. Find the square root of

$$9a^6-12a^5-26a^4+44a^3+9a^2-40a+16.$$

9. Find the square root of 494,209; of 57,686.4324.

10. What is a radical? Give three illustrations.

11. Write the radical expressing the 12th root of 125; the 30th root of 27; the nth root of a.

12. What is meant by the index of a radical? by the radicand?

13. Is $\sqrt{25}$ a radical? is $\sqrt[3]{25}$ a radical? Give reason in each case.

14. What Axiom is used to get rid of the radical in solving an equation containing a radical sign?

15. Solve the equations

$$\sqrt{2x+1}+7=x.$$
$$\sqrt{2x+5}-\sqrt{x-1}=2.$$
$$\sqrt{x+2}-\sqrt{2x-10}=\sqrt{3x-20}.$$

16. When is a radical in its simplest form?

17. What are similar radicals? Give illustrations.

18. Simplify $\sqrt{75}-4\sqrt{243}+6\sqrt{12}$.

19. What is a surd? Give illustrations.

20. Rationalize the denominators in

$$\frac{5}{\sqrt{2}},\ \frac{5}{\sqrt[3]{2}},\ \frac{3xy}{\sqrt{27x}},\ \frac{5}{\sqrt{x}+1},\ \frac{5}{\sqrt{3}-\sqrt{2}}.$$

21. What is a quadratic equation? How is it different from a linear equation? Give illustrations of each.

22. Define and illustrate a pure quadratic.

23. Define and illustrate an affected quadratic.

24. What is the principle used in solving quadratic equations by factoring?

25. What kind of quadratic requires that its square be "completed" before solving?

26. Describe accurately the method which you use for "completing the square" in solving quadratics.

27. Solve each of the following equations by factoring and check your answers.

(*a*) $x^2-5x+6=0.$

(*b*) $x^2-22x=23.$

(*c*) $x^2-2x=4x+55.$

(*d*) $x^2-\frac{7}{3}x-2=0.$

28. Solve each of the following equations by completing the square.

(*a*) $2x^2+3x-4=0$.
(*c*) $3x^2-x-10=0$.
(*b*) $7x^2-8x-9=0$.
(*d*) $\sqrt{x+1}+x=11$.

29. The solutions of the quadratic equation for a problem do not always both satisfy the problem itself. Explain this. Examine to see whether it is illustrated in the working of the following problem:

Find the *positive integer* such that the square of it added to six times the number gives 27.

30. Explain how one goes to work to draw the graph of a given quadratic equation.

31. In how many points must the graph of a quadratic cut the XX axis in case the two solutions are *real?* What happens to the graph in case the two solutions are *imaginary?* Give simple illustrations.

32. What especial interest attaches to the points mentioned in Ex. 31 in case such points are present?

33. Draw the graph of the quadratic $x^2-4x=0$.

34. The sum of two numbers is 29 and the sum of their squares is 505. Find the numbers.

35. A string is just long enough to go around a certain square. If 3 feet are cut off the string, it will then just reach round a square whose area is $\frac{4}{9}$ that of the first square. How long is the string?

For further exercises on this Chapter, see Appendix, p. 310.

APPENDIX

PART I. SUPPLEMENTARY EXERCISES

These exercises are intended as a supplement to those given in the body of the text. They may be used, at the discretion of the teacher, either for additional drill exercises during the first study of a topic, or for reviews.

Supplementary Exercises on § 3

1. Envelopes which cost 7 cents a package at wholesale are sold for 3 cents more a package at retail; what is the retail price?

2. If the envelopes cost c cents a package at wholesale and are sold at retail for r cents more, what represents the retail price?

3. What is the retail price in Ex. 2 if $c=6$ cents, and $r=3$ cents? if $c=5\frac{1}{2}$ cents, and $r=2$ cents?

4. One man walks 15 miles in one day and another man walks 3 miles more in the same time. How far did the second man walk in one day?

5. If the first man walked b miles and the other walked e miles more in the same time, how far did the second man walk in one day?

6. In Ex. 5, how far did the second man walk if $b=20$, and $e=6$? if $b=19$, and $e=3$?

7. How many minutes in 3 hours? in d hours? in e hours?

8. Give the expression representing r more than 5.

9. If $n=8$, what is the value of $16+n$? of $25+n$?

10. Give the expression representing 5 less than k.

11. If $c=17$, what is the value of $20-c$? of $36-c$?

12. What is the next even number after 22?

13. If x is an even number, what represents the next even number after it?

14. If a boy is 13 years old now, how old will he be in 3 years? If he is x years old now, how old will he be in 3 years?

15. If a man's present age is represented by x, what represents his age 10 years ago?

16. What is the cost of 6 railway tickets at 75 cents each? What is the cost of x railway tickets at 75 cents each?

17. If a suit of clothes costs 9 times as much as a hat, and if the hat costs k dollars, what represents the cost of the suit? What represents the cost of both?

18. A baseball team scores 27 runs in 9 innings. What was the average per inning?

19. A baseball team scores y runs in 9 innings. What was their average per inning?

20. If a team scores 6 runs in y innings, what is its average per inning?

21. If I spend 40 cents to-day and r cents to-morrow, how much shall I spend in the two days?

22. If one part of 10 is 7, what is the other part?

23. If one part of a is 3, what is the other part?

24. If one part of 10 is a, what is the other part?

25. If x is the greater part of a number and the difference between the parts is 4, what is the other part?

26. If y is the smaller part of a number and if the smaller part is 4 less than the larger, what is the larger?

27. What is the expression showing how much $6\,a$ exceeds 13?

28. John has 3 times as much money as James, and James has 4 times as much money as Harry. If Harry has x dollars, how much has each of the others?

SUPPLEMENTARY EXERCISES ON §§ 6–13

Solve each of the following equations.

1. $x+4=20$.

2. $x-6=32$.

3. $x+12=17$.

4. $2\,x-8=16$.

5. $3\,x+18=42$.

6. $5\,x-10=40$.

7. $2\,x=20-3\,x$.

8. $x=22-x$.

9. $3\,x-10=2\,x$.

10. $5\,x=21+2\,x$.

11. $x=35-6\,x$.

12. $4\,x-2\,x=18$.

13. $5\,x=29-4$.

14. $\frac{1}{3}\,x=10$.

15. $\frac{1}{2}\,x+2=8$.

16. $\frac{2}{3}\,x+6=18$.

17. $\frac{3}{4}\,x-12=6$.

18. $x+5\,x=8+10$.

19. $2\,x+5=6$.

20. $3\,x-4=18$.

SUPPLEMENTARY EXERCISES ON §§ 16–17

1. Give the negative (or antonym) of (*a*) sunrise, (*b*) to hoist the flag; (*c*) clean; (*d*) to increase; (*e*) overhead.

Add the following.

2. $\begin{array}{r} +16 \\ \underline{+\ 8} \end{array}$	**7.** $\begin{array}{r} +15 \\ \underline{+36} \end{array}$	**12.** $\begin{array}{r} -122 \\ \underline{+\ 22} \end{array}$	**17.** $\begin{array}{r} +72 \\ \underline{-15} \end{array}$
3. $\begin{array}{r} +32 \\ \underline{-\ 7} \end{array}$	**8.** $\begin{array}{r} +19 \\ \underline{-36} \end{array}$	**13.** $\begin{array}{r} +174 \\ \underline{-100} \end{array}$	**18.** $\begin{array}{r} +29 \\ \underline{-29} \end{array}$
4. $\begin{array}{r} -20 \\ \underline{+16} \end{array}$	**9.** $\begin{array}{r} +46 \\ \underline{+10} \end{array}$	**14.** $\begin{array}{r} -88 \\ \underline{-19} \end{array}$	**19.** $\begin{array}{r} -38 \\ \underline{-38} \end{array}$
5. $\begin{array}{r} -31 \\ \underline{-18} \end{array}$	**10.** $\begin{array}{r} +3.7 \\ \underline{-5.7} \end{array}$	**15.** $\begin{array}{r} -42 \\ \underline{-17} \end{array}$	**20.** $\begin{array}{r} +75 \\ \underline{-74} \end{array}$
6. $\begin{array}{r} -12 \\ \underline{+42} \end{array}$	**11.** $\begin{array}{r} -8.36 \\ \underline{+2.15} \end{array}$	**16.** $\begin{array}{r} -81 \\ \underline{+23} \end{array}$	**21.** $\begin{array}{r} -97 \\ \underline{+96} \end{array}$

SUPPLEMENTARY EXERCISES ON §§ 18–20

Find the value of each of the following expressions.

1. $6+(-2)+(-1)$.

2. $-4+(-3)+7$.

3. $9+(-8)+(-10)$.

4. $-6+(-7)+(-8)$.

5. $-18+12+(-4)$.

6. $17+25+(-42)$.

7. $-19+(-27)+(-13)$.

8. $36+(-19)+(-1)$.

9. $28+14+36$.

10. $-28+(-14)+(-36)$.

11. $\begin{array}{r} -37 \\ 26 \\ -18 \\ \underline{14} \end{array}$	**13.** $\begin{array}{r} -20 \\ 15 \\ 18 \\ \underline{-10} \end{array}$	**15.** $\begin{array}{r} -\ 1 \\ +14 \\ -36 \\ \underline{+23} \end{array}$	**17.** $\begin{array}{r} 126 \\ -132 \\ 134 \\ \underline{-130} \end{array}$	**19.** $\begin{array}{r} -42 \\ 36 \\ -28 \\ -15 \\ \underline{17} \end{array}$
12. $\begin{array}{r} 42 \\ 27 \\ -\ 9 \\ \underline{-15} \end{array}$	**14.** $\begin{array}{r} -15 \\ -18 \\ +31 \\ \underline{-\ 7} \end{array}$	**16.** $\begin{array}{r} 19 \\ 26 \\ 18 \\ \underline{-50} \end{array}$	**18.** $\begin{array}{r} 327 \\ 236 \\ -471 \\ \underline{-124} \end{array}$	**20.** $\begin{array}{r} 84 \\ -84 \\ 72 \\ -72 \\ 25 \end{array}$

21. How do you tell whether one negative number is greater or less than another one? Illustrate in the case of the numbers -11 and -2.

22. A frog by slow degrees jumps out of a well; it jumps upward 3 feet the first day and falls back 2 feet; 5 feet the second day and falls back 3 feet; 4 feet the third day and falls back 4 feet. How far is it from the bottom of the well at the end of the third day? Work by using *negative* numbers.

Supplementary Exercises on § 21

Carry out the following indicated operations.

1. $18-12$.
2. $21-(-3)$.
3. $-24-16$.
4. $-12-(-4)$.
5. $20-9$.
6. $-20-(-9)$.
7. $19-(-19)$.
8. $-19-(-19)$.
9. $40-(-10)$.
10. $36-18$.
11. $-81-(-80)$.
12. $-24-(-40)$.
13. $36-(-18)$.
14. $84-(-10)$.
15. $-10-6$.
16. $-78-(-78)$.
17. $26-(-26)$.
18. $32-50$.
19. $-76-50$.
20. $-82-(-31)$.

Supplementary Exercises on §§ 23–24

State the value of each of the following expressions.

1. $6 \cdot 4$.
2. $3(-9)$.
3. $7(-6)$.
4. $-8 \cdot 2$.
5. $-6(-3)$.
6. $10(-10)$.
7. $-15(-7)$.
8. $21 \cdot 12$.
9. $6 \cdot a$.
10. $-5 \cdot a$.
11. $10(-b)$.
12. $-10(-b)$.
13. $x \cdot y$.
14. $2x(-3y)$.
15. $-4x(-8y)$.
16. $-5x \cdot 5y$.
17. $3 \cdot 2 \cdot a$.
18. $-4(-2)b$.
19. $8(-2)(-b)$.
20. $-5(-4)(-x)$.
21. $(-a)(-b)(-c)(-d)$.
22. $abc(-rst)6$.
23. $3r(-2s)(-5t)$.
24. $a(-5b)(-2c)(-d)$.
25. $6r(-2s)(-3y)4b$.

SUPPLEMENTARY EXERCISES ON § 27

Carry out the following indicated operations.

1. $30 \div 3$.

2. $45 \div (-9)$.

3. $-65 \div 5$.

4. $-80 \div (-20)$.

5. $-38 \div (-2)$.

6. $76 \div (-9)$.

7. $-99 \div 11$.

8. $125 \div 25$.

9. $16\,a \div 4\,a$.

10. $-32\,x \div (-16\,x)$.

11. $-45\,x \div 9\,x$.

12. $72\,x \div (-36\,x)$.

13. $-20\,rs \div 4$.

14. $-36\,xy \div (-6)$.

15. $x \div y$.

16. $-x \div y$.

17. $-x \div (-y)$.

18. $x \div (-y)$.

19. $\frac{2}{3} \div (-\frac{2}{3})$.

20. $-4.6 \div (-2.3)$.

21. Solve by inspection each of the following equations.

(*a*) $-2\,x = 6$.

(*b*) $-2\,x = -6$.

(*c*) $3\,x = -4$.

(*d*) $-3\,x = 4$.

(*e*) $2\,x - 3 = 5$.

(*f*) $-2\,x + 3 = 5$.

(*g*) $3\,x - 2 = 7$.

(*h*) $-x - 1 = -1$.

(*i*) $-\dfrac{x}{2} + 3 = -5$.

(*j*) $-\dfrac{2}{x} = \dfrac{2}{3}$.

22. Find the value of each of the following expressions for the values written beside them.

(*a*) $\dfrac{a+b}{c+d}$, $[a=2,\ b=-1,\ c=-3,\ d=-1]$.

(*b*) $xyz \cdot (x+z)$, $[x=2,\ y=-3,\ z=1]$.

(*c*) $pq \div (p+q)$, $[p=3,\ q=-2]$.

(*d*) $\dfrac{x^2+y^2}{x+y} + \dfrac{x^2-y^2}{x-y}$, $[x=3,\ y=-2]$.

SUPPLEMENTARY EXERCISES ON § 29

Find the sum of the following.

1. $3\,a$ and $2\,a$.

2. $2\,k$ and $-3\,k$.

3. $-4\,c$ and $4\,c$.

4. $4\,x$ and $-2\,x$.

5. $3\,b$, $7\,b$, and $2\,b$.

6. $4\,a$, $-6\,a$, and $2\,a$.

7. $-4\,y$, $-2\,y$, and $-12\,y$.

8. $6\,a^2b$, $-6\,a^2b$, and $3\,a^2b$.

9. $7\,xyz^3$, $3\,xyz^3$, and $-9\,xyz^3$.

10. xy, $2\,xy$, $-4\,xy$, and $-7\,xy$.

SUPPLEMENTARY EXERCISES ON § 30

Simplify each of the following expressions.

1. $3a+2a-4a-5a$.
2. $7x-3x+2x-9x$.
3. $8x+12x-6x-2x$.
4. $6ab-15ab-7ab-2ab$.
5. $-4z+20z-15z-z$.
6. $3x^2y+2x^2y-8x^2y+3x^2y$.
7. $x^3y^3+x^3y^3-2x^3y^3+17x^3y^3$.
8. $2mny^2+12mny^2+10mny^2-23mny^2$.

9.
$$\begin{array}{r} -16\,xr^3 \\ 13\,xr^3 \\ -\;\;2\,xr^3 \\ -\;\;8\,xr^3 \\ 10\,xr^3 \\ \hline \end{array}$$

10.
$$\begin{array}{r} -25\,abc \\ 17\,abc \\ 2\,abc \\ -\;\;6\,abc \\ -19\,abc \\ \hline \end{array}$$

SUPPLEMENTARY EXERCISES ON § 32

Add the following.

1.
$$\begin{array}{l} 3x+2y \\ 3x-\;\;y \\ \hline \end{array}$$

2.
$$\begin{array}{l} 2r-7 \\ 5r+8 \\ \hline \end{array}$$

3.
$$\begin{array}{r} -5x+2 \\ 4x-7 \\ \hline \end{array}$$

4.
$$\begin{array}{r} -7a+2b+3c \\ 2a-5b-\;\;c \\ \hline \end{array}$$

5.
$$\begin{array}{r} 10r+2s+4t \\ -\;\;5r-2s-4t \\ \hline \end{array}$$

6.
$$\begin{array}{r} abc-\;\;rst+\;\;xyz-8 \\ 5abc+2rst-6xyz+10 \\ \hline \end{array}$$

7.
$$\begin{array}{r} 3a^2+2b^2+7c^2 \\ -5a^2-2b^2+8c^2 \\ -6a^2+3b^2-\;\;c^2 \\ \hline \end{array}$$

8.
$$\begin{array}{r} 5s+2s^2-3s^3 \\ -2s-5s^2+4s^3 \\ -5s-2s^2+3s^3 \\ \hline \end{array}$$

9.
$$\begin{array}{r} 2a+3x+4y+\;\;\;\;z \\ -5a-7x+2y-10z \\ 3a+4x-3y-\;\;2z \\ -\;\;a-3x+2y+\;\;2z \\ \hline \end{array}$$

10.
$$\begin{array}{rrrr} 8x & +3y & +4z & \\ 2x & -\;\;y & & -5k \\ -\;\;x & +2y & -2z & \\ -3x & -7y & +8z & +2k \\ \hline \end{array}$$

SUPPLEMENTARY EXERCISES ON § 35

Subtract.

1. $\begin{array}{r} 12\,a \\ \underline{-4\,a} \end{array}$

2. $\begin{array}{r} -8\,b \\ \underline{3\,b} \end{array}$

3. $\begin{array}{r} 10\,k \\ \underline{15\,k} \end{array}$

4. $\begin{array}{r} -6\,r \\ \underline{-4\,r} \end{array}$

5. $\begin{array}{r} 21\,a^2 \\ \underline{-13\,a^2} \end{array}$

6. $\begin{array}{r} -15\,rxy \\ \underline{-15\,rxy} \end{array}$

7. From $125\,x^2y$ take $89\,x^2y$.

8. Take $-13\,x^3$ from $-26\,x^3$.

9. $6\,a-(25\,a)=?$

10. $-14\,x-(-3\,x)=$

SUPPLEMENTARY EXERCISES ON § 36

Perform the following indicated subtractions.

1. $\begin{array}{r} 6\,x+4\,y \\ \underline{2\,x+7\,y} \end{array}$

2. $\begin{array}{r} 3\,x-2\,y \\ \underline{-3\,x-2\,y} \end{array}$

3. $\begin{array}{r} ab+2 \\ \underline{5\,ab+7} \end{array}$

4. From $a+3\,b$ take $10\,a-9\,b$.

5. Take $3\,x^2y-4$ from $10\,x^2y+9$.

6. Take $-4\,a+9\,b+7\,c$ from $10\,a-3\,b+14\,c$.

7. Subtract $a^3+3\,a^2b+3\,ab^2+b^3$ from $a^3-3\,a^2b+3\,ab^2-b^3$.

8. From $3\,xy+7\,z-9\,k$ take $8\,xy+4\,k-9\,z$.

9. From the sum of $3\,a^2+4\,a^3+2\,a-7\,a^4+1$ and $3+2\,a^2-4\,a+a^3-2\,a^4$ take $6\,a^3+2\,a-a^2-3\,a^4-9$.

10. How much larger is $6\,x^2+9\,x+10$ than $3\,x^2-2\,x-5$?

SUPPLEMENTARY EXERCISES ON § 37

1. State the four Axioms. Why is it that they are very useful in solving equations?

Solve the following equations and check each one.

2. $x+4=2$.
3. $x-10=3$.
4. $2+x=1$.
5. $18=3+x$.
6. $10=x+12$.
7. $12+7\,x=-9$.
8. $24=3\,x-3$.
9. $5\,x=-2.5$
10. $.3x=.81$
11. $-.1\,x=3$.
12. $2\,a-37=7\,a+3$.
13. $2\,r-15=9\,r-1$.
14. $10\,k+19=7\,k-12$.
15. $6\,x-(4\,x+3)=19$.
16. $3-(2\,x+8)=15\,x-(x-15)$.
17. If $6\,x-9=25$, what does $2\,x+11$ equal?

18. A rectangular field is three times as long as wide. The fence surrounding the field is 144 yards in length. Find the length and width of the field.

19. What is the number which increased by $\frac{3}{4}$ of itself equals 70?

20. The sum of three consecutive numbers is 93. What are the numbers?

21. If a number is increased by 42, the result is 45 more than twice the number. Find the number.

22. An employer on pay day wishes to have \$35 in change. He wishes to have as many quarter dollars as nickels, and twice as many dimes as nickels. How many must he get of each?

23. There are two rows of crosswise seats in a street car, and two long lengthwise seats. If each crosswise seat will hold two persons and each lengthwise seat will hold eight persons, how many crosswise seats must there be in order that the seating capacity of the car shall be 44?

24. If a quantity of grape juice is bottled in 5 bottles, 1 gill remains, but if it is bottled in 2 bottles twice as large, 4 gills remain. How many gills does each bottle contain?

25. A cask is $\frac{3}{4}$ full; $7\frac{1}{2}$ gallons are drawn from it and it is now half empty. How many gallons does the cask contain?

26. A person's income is \$1290; $\frac{2}{3}$ of his capital is invested at 4% and the remainder at 2%. Find the capital.

27. A man invests \$1000, part at 3% and part at 4%. The annual income from the investment is \$34. How has he divided the money?

28. 20 per cent of the pupils in a class drop out while 13 new pupils enter. The class is now found to be increased by $\frac{1}{3}$ its original number. How many were there at first?

29. Solve the equation $3(4-x)^2 - 2(x+3) = (2x-3)^2 - (x+2)(x-2) + 1$.

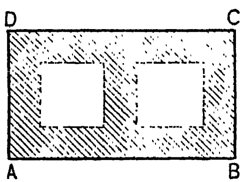


FIG. 93.

30. The figure represents a rectangle, $ABCD$, within which are placed two equal squares. The border (shaded) is everywhere 3 inches wide. In order that the area of the border be 159 square inches, what must be the length of one side of each square?

SUPPLEMENTARY EXERCISES ON § 42

Simplify each of the following expressions.

1. $5x-4y-(x-2y)+5x-(3x+2y)$.
2. $6a^2+3ac-2c^2-(-3a^2-2ac+c^2)$.
3. $x+y+z-(x+y-z)$.
4. $12b-(3b-\{-b+5\})$.
5. $6x-(x^2-\{x-5\}+4)$.
6. $10a-(-2a+7)-(6a+4)-10$.
7. $c^2-4-[4c^2+(-8-c^2)-2]$.
8. $-\{-[-(a+b)]\}$.
9. $a-\{3a-[-(-3a+2b)+5b]-3y\}$.
10. $2x-[x-\{y-(3y-3x)\}+4y]-(y-x)$.

SUPPLEMENTARY EXERCISES ON § 47

Solve each of the following equations for x.

1. $x+5=4x-1$.
2. $6x+1=3x+10$.
3. $4x-4=2x+6$.
4. $8-5x=20+5x$.
5. $\frac{3}{4}x-7=\frac{1}{4}x-27$.
6. $10x+8-2=3x-15$.
7. $2x-5+3x=15-10+2x$.
8. $3x-27=4x+10-27$.

9. If 16 is added to 4 times a certain number, the result is the difference between 106 and the number. Find the number.

10. Three boys have 195 marbles; if the second has 20 more than twice as many as the first, and the third has 35 more than the first, how many marbles has each?

11. If 18 is added to a certain number, the result is the same as 3 times the number subtracted from 346. What is the number?

12. Of three given numbers each exceeds the one below it by 4, and the sum of these numbers is 72. What are the numbers?

SUPPLEMENTARY EXERCISES ON § 51

Multiply

1. $r+s$ by 5.
2. $r-s$ by 3.
3. $r+7$ by -4.
4. $r+s$ by s.
5. r^2+s^2 by s^2.
6. $a^2+7\ ab-4\ b^2$ by $3\ a^2b^2$.
7. $2\ a^2-3\ ab+6\ b^2$ by $-4\ ab$.
8. $4\ r^2+2\ r^2b-5\ b^2$ by $2\ rb^2$.

Simplify

9. $r^2(2\ r+3\ r^2-7\ r^3)$.
10. $(x^2+3\ xy+2\ y^2)xy$.
11. $-rs(r^2s^2-2\ rs+5\ r)$.
12. $2(a+b)-2(a-b)$.
13. $4\ c(a+c)+2(ac-2\ c^2)$.
14. $r^2s^2-rs(rs-1)$.
15. $6\ a^2(1-b)+b(6\ b+3\ b^2)$.
16. $4\ x^2y^2-2\ xy(2xy-1)-2\ xy$.

SUPPLEMENTARY EXERCISES ON § 52

Factor each of the following expressions.

1. $6\ a+6\ b$.
2. $xy+xz$.
3. $xy-xz$.
4. $a+a^2$.
5. $ay+a^2y^2$.
6. $2\ x^2+6\ x^3+8\ x^4$.
7. $ab^2+a^2b^2-a^3b^3$.
8. $3\ rs^2-9\ r^2s^2y+12\ r^3sy^2$.
9. $15\ x^2y^2z+10\ xy^2z^2-5\ x^2yz^2$.
10. $35\ abc+70\ ab^3c^2-30\ a^3b^2c$.

SUPPLEMENTARY EXER

Multiply

1. $(x+a)(x-a)$.
2. $(a+b)(a+b)$.
3. $(a-b)(a-b)$.
4. $(a+6)(a-6)$.
5. $(a-2)(a+4)$.
6. $(2\ a+7)(3\ a-6)$.
7. $(3\ a+4\ b-6)(3\ a-4\ b+6)$.
8. $(6\ x^2-5\ x+7)(3\ x^2+2\ x-7)$.
9. $(2\ x+3\ y-4\ z)(4\ x-3\ y+2\ z)$.
10. $(10\ a+5\ b+4)(6\ a+3)$.

11. The side of a square is represented by $3\ a+4$. What represents its area?

12. The edge of a cube is represented by $x+y$. What represents its area?

13. Find the area of a triangle whose base is $3\ r-7$ and whose altitude is $2\ r+6$.

14. What represents the area of a trapezoid whose bases are $3\ x$ and $3\ y$ and whose altitude is $x+2\ y$?

15. What represents the area of a circle whose radius is $4-r$?

SUPPLEMENTARY EXERCISES ON § 56

Solve each of the following equations for x.

1. $6(x+2)=24$.
2. $3(x-7)=-42$.
3. $2(x+8)-(x+4)=42$.
4. $(x+6)(2x+4)=2(x^2+8)$.
5. $(x+10)(3x-5)=(3x+2)(x+4)$.
6. $(x+3)(5-x)+x(x+2)=0$.
7. $3(x+1)^2-2(x-1)^2=(x+6)^2-5(x+2)$.
8. $x^2+(x+7)(x-2)=2(x+3)(x-1)$.
9. $2x-1-(x+3)^2=4x-(x-2)(x+7)$.
10. $(2x^2-x+1)(x+3)=2x^2(x+2\frac{1}{2})-(10x-7)$.

SUPPLEMENTARY EXERCISES ON § 57

Multiply

1. $(x+5)(x+2)$.
2. $(m+2)(m-3)$.
3. $(y-5)(y-2)$.
4. $(r+7)(r-1)$.
5. $(b+9)(b-3)$.
6. $(a^2-3)(a^2+5)$.
7. $(ab+10)(ab-2)$.
8. $(H+1)(H-8)$.
9. $(x^2y^2+5)(x^2y^2+6)$.
10. $(3x+2y)(3x-4y)$.

SUPPLEMENTARY EXERCISES ON § 58

Factor each of the following expressions.

1. $r^2+7r+12$.
2. $r^2-3r-54$.
3. $x^2-14x+24$.
4. $x^2-4x-96$.
5. $s^2-13s+42$.
6. y^2+7y-8.
7. $x^2-13x-30$.
8. $4x^2-8xy-5y^2$.
9. $25r^2-10r-24$.
10. $9x^2-6xy-8y^2$.
11. $16m^2+4m-6$.
12. a^2x^2+ax-2.

SUPPLEMENTARY EXERCISES ON § 61

Expand each of the following expressions.

1. $(x+3)^2$.
2. $(x-3)^2$.
3. $(x+10)^2$.
4. $(x-7)^2$.
5. $(2x+1)^2$.
6. $(2x-3)^2$.

7. $(2x+2y)^2$.
8. $(3x-4y)^2$.
9. $(x^2y^2-z^2)^2$.
10. $(\frac{1}{2}ab+k)^2$.
11. $(.3x+.4y)^2$.
12. $[(a+b)+c]^2$.

SUPPLEMENTARY EXERCISES ON § 63

Factor each of the following expressions.

1. $a^2+2ab+b^2$.
2. $r^2-2rs+s^2$.
3. r^2+k^2-2rk.
4. $4a^2+4a+1$.
5. $36b^2-12ab+a^2$.
6. $x^2y^2-4xy+4$.
7. $4x^2+8x+4$.
8. $49a^2-28a+4$.
9. $144m^2-144m+36$.
10. $\frac{1}{4}m^2-\frac{1}{4}mn+\frac{1}{16}n^2$.

SUPPLEMENTARY EXERCISES ON § 64

Express as the square of some binomial in cases where it is possible.

1. x^2+4x+4.
2. x^2-4x+4.
3. $4r^2+4r+1$.
4. $16-8x+x^2$.
5. $64-32r+4r^2$.
6. $100n^2-160n+64$.
7. $81x^2y^2+72xy+16$.
8. $x^2+x+\frac{1}{4}$.
9. $\frac{1}{9}a^2-\frac{8}{3}a+16$.
10. $a^4+14a^2b+36b^2$.

SUPPLEMENTARY EXERCISES ON § 65

Multiply:

1. $(x+4)(x-4)$.
2. $(x-7)(x+7)$.
3. $(r+10)(r-10)$.
4. $(8+c)(8-c)$.
5. $(11-2a)(11+2a)$.
6. $(z+25)(z-25)$.
7. $(5k+r)(5k-r)$.
8. $(3ab+b)(3ab-b)$.
9. $(2x^2y-3)(2x^2y+3)$.
10. $(\frac{1}{2}x+\frac{2}{3}y)(\frac{1}{2}x-\frac{2}{3}y)$.

SUPPLEMENTARY EXERCISES ON § 67

Factor each of the following expressions.

1. a^2-25.
2. $100-c^2$.
3. k^2-81.
4. $81-k^2$.
5. $x^2-y^2z^2$.
6. $144k^2-1$.
7. $64k^2-49y^2$.
8. $\frac{25}{36}a^2-\frac{1}{49}$.
9. $x^2-(r+s)^2$.
10. $(2x+5y)^2-(a+b)^2$.

SUPPLEMENTARY EXERCISES ON § 68

Solve each of the following equations, finding *two* solutions for each.

1. $x^2-2x-15=0$.
2. $x^2-7x+12=0$.
3. $x^2-10x+25=0$.
4. $a^2-25=0$.
5. $x^2-x-20=0$.
6. $25x^2-20x+4=0$.
7. $\frac{1}{9}x^2-\frac{4}{3}x+4=0$.
8. $3x+28=x^2$.
9. $5x^2-25x=0$.
10. $\frac{1}{4}a^2-\frac{1}{2}a=0$.

SUPPLEMENTARY EXERCISES ON § 74

Carry out each of the following indicated divisions.

1. $(2x^2+11x+15)\div(x+3)$.
2. $(8x^2+18x-56)\div(4x-7)$.
3. $(4x^2-11x+3)\div(2x-3)$.
4. $(a^2-ab+a-b)\div(a+1)$.
5. $(2x^3+7x^2+11x-4)\div(2x^2+3x-1)$.

[HINT. Find both quotient and remainder.]

6. $(a^4-16)\div(a-2)$.
7. $(x^3-1)\div(x-1)$.
8. $(x^6-y^6)\div(x^2-y^2)$.
9. $(4x^2+3x-2)\div(2x+1)$.
10. $(x^2-x-10)\div(x-5)$.

SUPPLEMENTARY EXERCISES ON § 94

Simplify each of the following expressions.

1. $\frac{10x^2y^3}{15x^3y^5}$.
2. $\frac{27r^2s^5}{45r^6s^2}$.
3. $\frac{-19abc}{38abc}$.
4. $\frac{72xr^3}{-84x^3r}$.
5. $\frac{a-5}{a^2-10a+25}$.
6. $\frac{x^2-12x+11}{x-1}$.
7. $\frac{25a^2-16b^2}{25a^2+40ab+16b^2}$.
8. $\frac{ax+ay}{x^2+2xy+y^2}$.
9. $\frac{a^2-3a-54}{81-18a+a^2}$.
10. $\frac{r^3-7r^2-60r}{r^2-12r}$.

SUPPLEMENTARY EXERCISES ON § 99

Simplify each of the following expressions.

1. $\frac{x+1}{3}+\frac{x-1}{2}$.

2. $\frac{x+4}{12}-\frac{2x+4}{18}$.

3. $k-\frac{2(6-k)}{3}$.

4. $a+b-\frac{7(a-b)}{6}$.

5. $\frac{2}{3}(a+b)-\frac{1}{5}(a-b)$.

6. $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}$.

7. $\frac{1}{a+b}+\frac{1}{a^2+2ab+b^2}$.

8. $\frac{x+3}{x^2-6x+8}-\frac{2-x}{x-4}$.

9. $\frac{1-r}{r^2+8r+7}+\frac{2}{r+7}-\frac{6}{r+1}$.

10. $\frac{2}{x^2+8x-20}+\frac{5}{x^2+4x-12}$.

11. $\frac{a+1}{a-1}+\frac{2}{a+1}+\frac{4a}{1-a^2}$.

12. $\frac{1}{x}+1+\frac{2x}{1+x}-2$.

13. $\frac{a+x}{a-x}+\frac{a^2+x^2}{a^2-x^2}-\frac{a-x}{a+x}-\frac{a^2-x^2}{a^2+x^2}-\frac{4a^3x+4ax^3}{a^4-x^4}$.

14. $\frac{5(x-3)}{x^2-x-2}-\frac{2(x+2)}{x^2+4x+3}-\frac{x-1}{6-x-x^2}$.

SUPPLEMENTARY EXERCISES ON §§ 100–101

Carry out each of the following indicated multiplications.

1. $\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{4}{5}$.

2. $-\frac{a}{6}\cdot\frac{4a}{10}\cdot\frac{15}{a}$.

3. $\frac{3x}{y}\cdot\frac{3x}{2y}\cdot\frac{y}{2x}$.

4. $\frac{a+b}{a^2-b^2}\cdot\frac{a-b}{a+b}$.

5. $\frac{6r-6s}{r^2-8rs+16s^2}\cdot\frac{r-4s}{3(r-s)}$.

6. $a\cdot\frac{a}{b}\cdot\frac{b}{a}$.

7. $(r+k)\cdot\frac{r-k}{r^2-k^2}\cdot\frac{r-k}{6}$.

8. $\frac{(x-5)(x-4)}{x-2}\cdot\frac{(x-2)(x-1)}{x-4}\cdot\frac{1}{(x-5)(x-1)}$.

9. $\frac{a-7}{a+3}\cdot\frac{9-a^2}{49-a^2}$.

10. $\frac{x^2+x-2}{x^2-5x+4}\cdot\frac{x^2+x-20}{x^2-2x-8}\cdot\frac{x^2+4x-5}{x^2+7x+10}$.

SUPPLEMENTARY EXERCISES ON § 102

Carry out each of the following indicated divisions.

1. $\frac{7a^2}{9bc} \div a^2.$

2. $\frac{6x^2y}{wv} \div 6xy.$

3. $\frac{3x^2y}{4ab^3} \div \frac{2x}{3ab}.$

4. $\frac{x^2-16}{a^2+2ab+b^2} \div \frac{x+4}{a+b}.$

5. $\frac{x^2-15x+36}{x-12} \div \frac{x-3}{10}.$

6. $\frac{x^2-9x+14}{x^2+9x+20} \div \frac{x^2+x-56}{x^2-x-20}.$

7. $\left(6+\frac{4}{ab}\right) \div \left(3-\frac{2}{ab}\right).$

8. $\frac{3ab+6b^2}{a^2-4} \div \frac{2ab+4b^2}{a+2}.$

9. $\frac{16x^2-25y^2}{4x-4y} \div (4x+5y).$

10. $\frac{(a+b)^2}{(a-b)^2} \div \frac{a^2-b^2}{a-b}.$

11. $\left(y-x+\frac{x^2}{y}\right) \div \left(\frac{x}{y^2}+\frac{y}{x^2}\right).$

12. $(a+c) \div \left(\frac{a^2-c^2}{1+x} \div \frac{a-c}{1-x^2}\right).$

13. Perform the indicated operations in the following expression and reduce your answer to its simplest form:

$$\left(\frac{m-3n}{m+n}\right)\left(1+\frac{4}{m+n}\right) \div \left(\frac{m}{n}+2-\frac{15n}{m}\right).$$

SUPPLEMENTARY EXERCISES ON §§ 105–106

Solve each of the following equations.

1. $\frac{6x}{7}+24=-12.$

2. $2x-\frac{3x}{4}=10.$

3. $\frac{2x}{3}+\frac{3x}{4}=6.$

4. $\frac{3x}{4}-\frac{x}{2}+\frac{2x}{5}=20.$

5. $x+\frac{x}{4}+\frac{3x}{5}=37.$

6. $\frac{x+3}{6}-\frac{1}{3}=\frac{12-x}{4}.$

7. $\frac{x-5}{3}-4=\frac{x+5}{3}-\frac{x-2}{5}$

8. $\frac{10}{x}-\frac{1}{4}-\frac{6}{5x}=\frac{1}{10}.$

9. $\frac{6}{x+10}=12.$

10. $\frac{1}{x+2}-\frac{5}{3(x+2)}=\frac{1}{9}.$

11. $\frac{2}{x-1}=\frac{5}{3x+1}.$

12. $\frac{2}{x-3}-\frac{3}{x-4}=-\frac{1}{x-1}.$

13. $\frac{4x+3}{9}=\frac{8x+19}{18}-\frac{7x-29}{5x-12}$.

14. $\frac{3x-4}{4}-\left(\frac{4x}{5}+\frac{x+2}{2}\right)=\frac{9x}{10}-\left(19+\frac{x+4}{2}\right)$.

15. $\frac{a-b+c}{x+a}=\frac{b-a+c}{x-a}$

16. The sum of two numbers is 39. If the greater number be divided by 1 more than the smaller, the quotient is 9. What are the numbers?

17. An electrician can wire a house in 8 days if he works alone; but with his helper he can do the work in 2 days. How long would it take the helper if he worked alone?

18. A cistern receives water from three pipes. The first can fill it in $3\frac{1}{2}$ hours, the second in 4 hours, and the third in 5 hours. How long will it take if the three run at the same time?

19. A can do a piece of work in 8 days, B in 6 days, and C in $5\frac{1}{2}$ days. How long will it take them working together?

20. If A can do a piece of work in c hours and B can do it in d hours, how long will it take them working together?

21. Separate 642 into two parts such that the greater divided by the less gives 20 for the quotient and 12 for the remainder.

22. A bicyclist and a pedestrian start at the same time for a place 55 miles away, the former traveling 4 times as fast as the latter. The bicyclist reaches the place and starts back, meeting the pedestrian $5\frac{1}{2}$ hours after they started. Find the rate of each.

23. An express train whose rate is 40 miles per hour starts 1 hour and 4 minutes after a freight train and overtakes it in 1 hour and 36 minutes. What is the rate of the freight train per hour?

24. It took a passenger train 175 feet long $7\frac{1}{2}$ seconds to pass completely a freight train 485 feet long, moving in the opposite direction. If the passenger train was traveling three times as fast as the freight train, find the rate of each per hour.

25. In an alloy of 90 ounces of silver and copper there are 6 ounces of silver. How much copper must be added that 10 ounces of the new alloy may contain $\frac{2}{5}$ of an ounce of silver?

SUPPLEMENTARY EXERCISES ON § 107

1. If the outer and inner diameters of a pipe are D inches and d inches respectively, and the length of pipe is 1 foot, show that the number of cubic feet of material in it is given by the formula

$$\frac{\pi l}{576}(D-d)(D+d) \text{ cu. ft.}$$

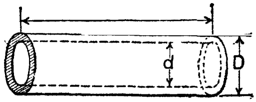


FIG. 94.

[HINT. The volume of a right cylinder is found by multiplying the area of its base by the height.]

2. In the figure are two equal-sized elastic balls resting upon a smooth table. Suppose ball 1 weighs W_1 (read W *sub* 1) ounces and that ball 2 weighs W_2 ounces, and suppose that ball 1 moves up with velocity of V_1 feet per second and hits (impinges upon) ball 2. Both balls will (in general) then move forward, and it may be shown that ball 1 will have had its velocity reduced to a value v_1, given by the formula

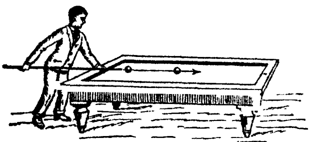


FIG. 95.

$$v_1 = \frac{W_1 - W_2}{W_1 + W_2} V_1 \text{ ft. per sec.}$$

while ball 2 will be found to have taken on a velocity v_2 given by the formula

$$v_2 = \frac{2\,W_1}{W_1 + W_2} V_1 \text{ ft. per sec.}$$

Show that when the two balls are made of the same material, and hence weigh the same (as for example, two ivory billiard balls) the effect of the striking (impact) is to simply bring ball 1 to rest; that is, reduce its velocity to zero, while the effect upon ball 2 is to make it move forward with precisely the velocity which 1 had in the first place.

3. Show that in the experiment described in Ex. 2, if ball 2 weighs more than ball 1, then the velocity of ball 1 after impact will be *negative*. The significance of this is that ball 1 will then bound *backward* instead of proceeding forward.

Show that if $W_1=2$ ounces, $W_2=10$ ounces, and $V_1=24$ feet per

second, then ball 1 will bound backward with a velocity of 16 feet per second.

4. Show that the total surface of a rectangular solid whose dimensions are a, b, and c is given by the formula

$$S=2\,a(b+c)+2\,bc.$$

5. If the inside measurements of an ice cream cone are *height* $= h$ inches, *diameter of base* $= d$ inches, show that the number N of such cones that can be filled from a cylindrical freezer H feet high and of diameter D feet is given by the formula

$$N=5184\,\frac{HD^2}{hd^2}.$$

[HINT. The volume of a cylinder is equal to the area of its base multiplied by its height, while the volume of a cone is equal to $\frac{1}{3}$ the product of the area of its base by its height.]

6. If the radius of the base of one hemisphere is twice that of another, what is the ratio of their areas? of their volumes?

7. The velocity at which sound travels is given by the formula,

$$v=1090+1.14(t-32)$$

where v is the velocity in feet per second and t the temperature of the air in Fahrenheit degrees. Find (1) the velocity of sound at a temperature of 82° and (2) the temperature of the air at which sound travels 2237 feet in 2 seconds.

SUPPLEMENTARY EXERCISES ON §§ 108–116

1. Is it true that $\frac{9}{12}=\frac{15}{20}$? Why?

2. Do 3, 4, 6, and 8 form a proportion taken in the order in which they stand?

3. If $r:18=(r-1):12$, find r.

4. Find the values of the letters in each of the following equations.

(a) $\dfrac{4}{3}=\dfrac{5}{\sqrt{x}}$. (b) $\dfrac{x^2}{16}=\dfrac{25}{49}$. (c) $\dfrac{\sqrt[n]{x}}{b}=\dfrac{c}{d}$.

5. Separate 48 into two parts, such that one part shall be to the other as 11 is to 13.

6. Divide \$475 between A and B so that A receives \$11 to B's \$8.

7. How many ounces of tin and copper, each, in 56 ounces of gun metal, if gun metal consists of 1 part of tin and 9 parts of copper?

8. The sides of a triangle are 8, 12, and 16 inches; the shortest side of a similar triangle is 10 inches. Find the remaining sides of the similar triangle.

NOTE. In two similar triangles any two sides have the same ratio as the corresponding sides of the other.

9. The diameter of the standard baseball is 3 inches. How does its surface compare with the surface of a slate globe 2 ft. in diameter?

NOTE. The areas of similar figures are to each other as the squares on corresponding lines.

10. What proportion exists between the times t_1 and t_2 it will take to do a certain piece of work if the number of men employed in the first case is n_1 and in the second case is n_2?

11. What proportion exists between the numbers of sheep that can be grazed in two square fields, one having a side of length a and the other a side of length b? Use n_1 and n_2 to represent the two numbers.

SUPPLEMENTARY EXERCISES ON §§ 123–124

Draw the graph of each of the following equations.

1. $x+8y=6$.

2. $x-3y=6$.

3. $2x-3y=12$.

4. $4x+3y=12$.

5. $-4x+3y=-12$.

6. $5x+y=4$.

SUPPLEMENTARY EXERCISES ON § 125

Solve each of the following simultaneous equations by first drawing their graphs.

1. $\begin{cases} x+y=10, \\ x-y=4. \end{cases}$

$\begin{cases} x+2y=6, \\ 2x+4y=10. \end{cases}$

$\begin{cases} 2x+y=6, \\ x+5y=-6. \end{cases}$

4. $\begin{cases} 3x-5y=12, \\ 6x-10y=24. \end{cases}$

5. $\begin{cases} 3x=y, \\ 5y-6x=18. \end{cases}$

6. $\begin{cases} x-2y=-3, \\ 2x+y=3. \end{cases}$

SUPPLEMENTARY EXERCISES ON §§ 128–130

Solve (by any method) the following simultaneous equations.

1. $\begin{cases} 3x-5y=-31, \\ 5x-3y=-9. \end{cases}$

2. $\begin{cases} 2x+5y=-8, \\ 2x-3y=0. \end{cases}$

3. $\begin{cases} x+y=-18, \\ 2x-3y=9. \end{cases}$

4. $\begin{cases} 8x-y=-9, \\ 2x+y=3. \end{cases}$

5. $\begin{cases} \frac{1}{2}x+2y=-12, \\ \frac{3}{4}x-3y=-16. \end{cases}$

6. $\begin{cases} 5x+6y=7, \\ 2x-5y=-49. \end{cases}$

7. $\begin{cases} 7x+3y=8, \\ \frac{1}{3}x+\frac{1}{2}y=\frac{13}{21}. \end{cases}$

8. $\begin{cases} 5x=4y, \\ 3x-2y=6. \end{cases}$

9. $\begin{cases} .1x+.3y=5.8, \\ .6x-.5y=-2. \end{cases}$

10. $\begin{cases} \dfrac{x+2}{3}-2y=28, \\ \dfrac{y+9}{5}+2x=25. \end{cases}$

11. $\begin{cases} \dfrac{x-18}{15}-4y=6, \\ \dfrac{x+14}{2}-\dfrac{3y+10}{4}=0. \end{cases}$

12. $\begin{cases} \dfrac{x}{5}+\dfrac{y}{10}=0, \\ 2x+\dfrac{3y-2x}{4}=0. \end{cases}$

13. $\begin{cases} \dfrac{x}{2}+\dfrac{16-x}{2}=30+\dfrac{5y+2x}{40-x}, \\ \dfrac{4(x-6)}{y+8}+\dfrac{83-8y}{8}=10-y. \end{cases}$

14. $\begin{cases} \dfrac{1}{x-a}=\dfrac{1}{a-y}, \\ \dfrac{x+y}{x-y}=a. \end{cases}$

15. A workman is engaged for 15 days; he receives \$2 for every day he works and agrees to forfeit 50 cents a day for every day he is idle. At the end of the time he receives \$22.50. How many days did he work?

16. Two boys, A and B, wishing to determine their weights, find that they balance on a teeter board when B is 6 feet from the fulcrum and A is 5 feet from it. If B carries a 30-pound weight with him, they balance when B is 4 feet and A 5 feet from the fulcrum. How heavy is each boy?

SUPPLEMENTARY EXERCISES ON § 136

Reduce to simpler form each of the following expressions.

1. $\sqrt{64}$. **2**. $\sqrt{a^8c^6}$. **3**. $\sqrt{900r^2s^6}$. **4**. $\sqrt{a^{2n}}$. **5**. $\sqrt{25a^6b^4c^{10}}$.

Find the square root of each of the following expressions.

6. $25x^2-80xy+64y^2$. **7**. $16x^2-2xy+\frac{1}{16}y^2$.

8. $4x^2+9y^2+16-12xy+16x-24y$.

9. $9\,a^2-24\,ab-12\,ac+16\,b^2+16\,bc+4\,c^2$.

10. $4+a^2-4\,a+2\,c-ac+\dfrac{c^2}{4}$.

11. $4\,a^2-12\,ab+16\,ac+9\,b^2+16\,c^2-24\,bc$.

12. $\dfrac{a^4}{4}+a^3x+\dfrac{4\,a^2x^2}{3}+\dfrac{2\,ax^3}{3}+\dfrac{x^4}{9}$.

SUPPLEMENTARY EXERCISES ON § 138

Solve each of the following equations. Check each answer.

1. $\sqrt{x^2+9}=5$.
2. $\sqrt{x^2-36}-8=0$.
3. $\sqrt{2\,x-2}=\sqrt{2\,x+10}-2$.
4. $\sqrt{4\,x-8}+\sqrt{4\,x+12}=10$.
5. $\sqrt{x}-\sqrt{12+x}=-2$.
6. $3\,x-\sqrt{9\,x^2-12\,x-51}=3$.
7. $x-1-\sqrt{x^2+x-11}=0$.
8. $\dfrac{6-\sqrt{x}}{\sqrt{x}}+1=\frac{3}{4}$.

SUPPLEMENTARY EXERCISES ON § 148

Rationalize the denominator in each of the following expressions.

1. $\dfrac{\sqrt{5}}{\sqrt{10}}$.
2. $\dfrac{8}{\sqrt{15}}$.
3. $\dfrac{r}{\sqrt{x}}$.
4. $\dfrac{\sqrt{2\,r}}{\sqrt{5\,r}}$.
5. $\dfrac{3\,ab}{\sqrt{6\,ax}}$.
6. $\dfrac{5}{\sqrt{5}-\sqrt{2}}$.
7. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.
8. $\dfrac{2+\sqrt{6}}{5\sqrt{3}}$.
9. $\dfrac{2\sqrt{a}-3\sqrt{b}}{\sqrt{a}-2\sqrt{b}}$.
10. $\dfrac{\sqrt{x-2}+4}{\sqrt{x-2}-3}$.

SUPPLEMENTARY EXERCISES ON § 152

Solve each of the following equations, finding the two roots.

1. $6\,x^2=216$.
2. $4\,x^2-36=0$.
3. $2\,x^2-\frac{9}{2}=0$.
4. $12\,x^2-10=7\,x^2+15$.
5. $(x+4)^2+(x-4)^2=32$.
6. $\dfrac{x-25}{x+25}=\dfrac{1-x}{1+x}$.
7. $\dfrac{x^2+2}{2}=5$.
8. $\dfrac{2\,x}{5}=\dfrac{5}{2\,x}$.
9. $x=\dfrac{r^2}{x}$.
10. $ax^2+c=b$.

SUPPLEMENTARY EXERCISES ON § 155

Solve each of the following quadratics by completing the square.

1. $x^2-4x=12$.
2. $x^2+6x=16$.
3. $x^2-10x+21=0$.
4. $x(x-6)=-9$.
5. $x(x+18)=-81$.
6. $49+x^2=14x$.
7. $24x+119=-x^2$.
8. $-\dfrac{120}{x+22}=x$.
9. $x^2+2bx=0$.
10. $4x^2-7=-12x$.

SUPPLEMENTARY EXERCISES ON §§ 156–157

Solve each of the following equations.

1. $9x^2+12x=12$.
2. $9x^2-6x=8$.
3. $25x^2-40x=9$.
4. $25x^2-20x+3=0$.
5. $16x^2-40x=0$.
6. $36x^2+4x=\frac{1}{3}$.
7. $3x^2-\frac{2}{3}x=\frac{8}{27}$.
8. $3x^2-4x=7$.
9. $27x^2-4x=4\frac{1}{3}$.
10. $\frac{1}{4}x^2-3x=27$.

11. If a wire l feet long is stretched between two poles of equal height which are H feet apart, the sag d, which the wire has at its middle point is given by the formula

$$d=\sqrt{\frac{3Hl-3H^2}{8}}\text{ ft.}$$

By use of this formula find the actual length of wire needed to stretch between two poles which are 150 feet apart provided a sag of 5 feet is to be allowed. $l=150.44^{+}$ ft. *Ans.*

12. The figure represents an elevated water tank such as is commonly used at manufacturing plants. The lower part consists of a hemispherical bowl upon which rests a cylindrical part and at the top is a conical roof. The water occupies the bowl and the cylinder, but not the cone. If the height of the cylinder be $1\frac{1}{2}$ times its diameter, what must be the diameter in order that the tank may hold 24,200 gallons? (Work in feet, allowing .14 cubic foot to a gallon and use the tables for the extraction of necessary cube roots.)

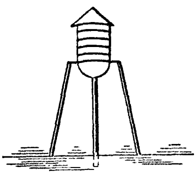


FIG. 96.

$2\sqrt[3]{294}=13.2988$ ft. *Ans.*

13. The sum of the areas of two circles is A and the sum of their radii is s. Find the formulas giving the radius of each.

PART II. TABLE OF POWERS AND ROOTS

Explanation

1. Square Roots. The way to find square roots from the Table is best understood from an example. Thus, suppose we wish to find $\sqrt{1.48}$. To do this we first locate 1.48 in the column headed by the letter $\boldsymbol{n}$. We find it near the bottom of this column (next to the last number). Now we go across on that level until we get into the column headed by $\sqrt{\boldsymbol{n}}$. We find at that place the number 1.21655. This is our answer. That is, $\sqrt{1.48}=1.21655$ (approximately).

If we had wanted $\sqrt{14.8}$ instead of $\sqrt{1.48}$ the work would have been the same except that we would have gone over into the column headed $\sqrt{\mathbf{10}\,\boldsymbol{n}}$ (because $14.8=10\times1.48$). The number thus located is seen to be 3.84708, which is, therefore, the desired value of $\sqrt{14.8}$.

Again, if we had wished to find $\sqrt{148}$ the work would take us back again to the column headed $\sqrt{\boldsymbol{n}}$, but now instead of the answer being 1.21655 it would be 12.1655. In other words, the order of the digits in $\sqrt{148}$ is the same as for $\sqrt{1.48}$, but the decimal point in the answer is one place farther to the right.

Similarly, if we desired $\sqrt{1480}$ the work would be the same as before except that we must now use the column headed $\sqrt{\mathbf{10}\,\boldsymbol{n}}$ and move the decimal point there occurring one place farther to the right. This is seen to give 38.4708.

Thus we see how to get the square root of 1.48 or any power of 10 times that number.

In the same way, if we wish to find $\sqrt{.148}$, or $\sqrt{.0148}$, or $\sqrt{.00148}$, or the square root of any number obtained by dividing 1.48 by any power of 10, we can get the answers from the column headed $\sqrt{n}$ or $\sqrt{10\,n}$ by merely placing the decimal point properly. Thus, we find that $\sqrt{.148} = .384708$, $\sqrt{.0148} = .121655$, $\sqrt{.00148} = .0384708$, etc.

What we have seen in regard to the square root of 1.48 or of that number multiplied or divided by any power of 10 holds true in a similar way for *any* number that occurs in the column headed n, so that the tables thus give us the square roots of a great many numbers.

2. Cube Roots. Cube roots are located in the tables in much the same way as that just described for square roots, but we have here three columns to select from instead of two, namely the columns headed $\sqrt[3]{n}$, $\sqrt[3]{10\,n}$, $\sqrt[3]{100\,n}$.

Illustration.

$\sqrt[3]{1.48}$ occurs in the column headed $\sqrt[3]{n}$ and is seen to be 1.13960.
$\sqrt[3]{14.8}$ occurs in the column headed $\sqrt[3]{10\,n}$ and is seen to be 2.4552.
$\sqrt[3]{148}$ occurs in the column headed $\sqrt[3]{100\,n}$ and is seen to be 5.28957.

To get $\sqrt[3]{.148}$ we observe that $.148 = \sqrt[3]{\frac{1.48}{10}} = \sqrt[3]{\frac{148}{1000}} = \frac{1}{10}\sqrt[3]{148}$. Thus, we look up $\sqrt[3]{148}$ and divide it by 10. The result is instantly seen to be .528957. Similarly, to get $\sqrt[3]{.0148}$ we observe that $\sqrt[3]{.0148} = \sqrt[3]{\frac{1.48}{100}} = \sqrt[3]{\frac{14.8}{1000}} = \frac{1}{10}\sqrt[3]{14.8}$. Thus, we look up $\sqrt[3]{14.8}$ and divide it by 10, giving the result .24552.

To get $\sqrt[3]{.00148}$ we observe that $\sqrt[3]{.00148} = \sqrt{\frac{1.48}{1000}} = \frac{1}{10}\sqrt{1.48}$, so that we must divide $\sqrt{1.48}$ by 10. This gives .11396.

Similarly the cube root of any number occurring in the column headed n may be found, as well as the cube root of any number obtained by multiplying or dividing such a number by any power of 10.

3. Squares and Cubes. To find the square of 1.48 we naturally look at the proper level in the column headed n^2. Here we find 2.1904, which is the answer. If we wished the square of 14.8 the result would be the same except that the decimal point must be moved *two* places to the *right*, giving 219.04 as the answer. Similarly the value of $(148)^2$ is 21904.0 etc.

On the other hand, the value of $(.148)^2$ is found by moving the decimal place two places to the *left*, thus giving .021904. Similarly, $(.0148)^2 = .00021904$, etc.

To find $(1.48)^3$ we look at the proper level in the column headed n^3 where we find 3.24179. The value of $(14.8)^3$ is the same except that we must move the decimal point *three* places to the *right*, giving 3241.79. Similarly, in finding $(.148)^3$ we must move the decimal place three places to the *left*, giving .00324179.

Further illustrations of the way to use the tables will be found in § 140.

EXERCISES

Read off from the tables the values of each of the following expressions.

1. $\sqrt{41}$	**4.** $\sqrt[3]{670}$	**7.** $\sqrt{93.7}$	**10.** $\sqrt[3]{.00154}$
2. $\sqrt{8.9}$	**5.** $\sqrt{.89}$	**8.** $\sqrt[3]{93.7}$	**11.** $\sqrt{.000143}$
3. $\sqrt[3]{67}$	**6.** $\sqrt{.016}$	**9.** $\sqrt{.00154}$	**12.** $\sqrt[3]{.000143}$

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.00	1.0000	1.00000	3.16228	1.00000	1.00000	2.15443	4.64159
1.01	1.0201	1.00499	3.17805	1.03030	1.00332	2.16159	4.65701
1.02	1.0404	1.00995	3.19374	1.06121	1.00662	2.16870	4.67233
1.03	1.0609	1.01489	3.20936	1.09273	1.00990	2.17577	4.68755
1.04	1.0816	1.01980	3.22490	1.12486	1.01316	2.18279	4.70267
1.05	1.1025	1.02470	3.24037	1.15762	1.01640	2.18976	4.71769
1.06	1.1236	1.02956	3.25576	1.19102	1.01961	2.19669	4.73262
1.07	1.1449	1.03441	3.27109	1.22504	1.02281	2.20358	4.74746
1.08	1.1664	1.03923	3.28634	1.25971	1.02599	2.21042	4.76220
1.09	1.1881	1.04403	3.30151	1.29503	1.02914	2.21722	4.77686
1.10	1.2100	1.04881	3.31662	1.33100	1.03228	2.22398	4.79142
1.11	1.2321	1.05357	3.33167	1.36763	1.03540	2.23070	4.80590
1.12	1.2544	1.05830	3.34664	1.40493	1.03850	2.23738	4.82028
1.13	1.2769	1.06301	3.36155	1.44290	1.04158	2.24402	4.83459
1.14	1.2996	1.06771	3.37639	1.48154	1.04464	2.25062	4.84881
1.15	1.3225	1.07238	3.39116	1.52088	1.04769	2.25718	4.86294
1.16	1.3456	1.07703	3.40588	1.56090	1.05072	2.26370	4.87700
1.17	1.3689	1.08167	3.42053	1.60161	1.05373	2.27019	4.89097
1.18	1.3924	1.08628	3.43511	1.64303	1.05672	2.27664	4.90487
1.19	1.4161	1.09087	3.44964	1.68516	1.05970	2.28305	4.91868
1.20	1.4400	1.09545	3.46410	1.72800	1.06266	2.28943	4.93242
1.21	1.4641	1.10000	3.47851	1.77156	1.06560	2.29577	4.94609
1.22	1.4884	1.10454	3.49285	1.81585	1.06853	2.30208	4.95968
1.23	1.5129	1.10905	3.50714	1.86087	1.07144	2.30835	4.97319
1.24	1.5376	1.11355	3.52136	1.90662	1.07434	2.31459	4.98663
1.25	1.5625	1.11803	3.53553	1.95312	1.07722	2.32079	5.00000
1.26	1.5876	1.12250	3.54965	2.00038	1.08008	2.32697	5.01330
1.27	1.6129	1.12694	3.56371	2.04838	1.08293	2.33311	5.02653
1.28	1.6384	1.13137	3.57771	2.09715	1.08577	2.33921	5.03968
1.29	1.6641	1.13578	3.59166	2.14669	1.08859	2.34529	5.05277
1.30	1.6900	1.14018	3.60555	2.19700	1.09139	2.35133	5.06580
1.31	1.7161	1.14455	3.61939	2.24809	1.09418	2.35735	5.07875
1.32	1.7424	1.14891	3.63318	2.29997	1.09696	2.36333	5.09164
1.33	1.7689	1.15326	3.64692	2.35264	1.09972	2.36928	5.10447
1.34	1.7956	1.15758	3.66060	2.40610	1.10247	2.37521	5.11723
1.35	1.8225	1.16190	3.67423	2.46038	1.10521	2.38110	5.12993
1.36	1.8496	1.16619	3.68782	2.51546	1.10793	2.38697	5.14256
1.37	1.8769	1.17047	3.70135	2.57135	1.11064	2.39280	5.15514
1.38	1.9044	1.17473	3.71484	2.62807	1.11334	2.39861	5.16765
1.39	1.9321	1.17898	3.72827	2.68562	1.11602	2.40439	5.18010
1.40	1.9600	1.18322	3.74166	2.74400	1.11869	2.41014	5.19249
1.41	1.9881	1.18743	3.75500	2.80322	1.12135	2.41587	5.20483
1.42	2.0164	1.19164	3.76829	2.86329	1.12399	2.42156	5.21710
1.43	2.0449	1.19583	3.78153	2.92421	1.12662	2.42724	5.22932
1.44	2.0736	1.20000	3.79473	2.98598	1.12924	2.43288	5.24148
1.45	2.1025	1.20416	3.80789	3.04862	1.13185	2.43850	5.25359
1.46	2.1316	1.20830	3.82099	3.11214	1.13445	2.44409	5.26564
1.47	2.1609	1.21244	3.83406	3.17652	1.13703	2.44966	5.27763
1.48	2.1904	1.21655	3.84708	3.24179	1.13960	2.45520	5.28957
1.49	2.2201	1.22066	3.86005	3.30795	1.14216	2.46072	5.30146

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.50	2.2500	1.22474	3.87298	3.37500	1.14471	2.46621	5.31329
1.51	2.2801	1.22882	3.88587	3.44295	1.14725	2.47168	5.32507
1.52	2.3104	1.23288	3.89872	3.51181	1.14978	2.47712	5.33680
1.53	2.3409	1.23693	3.91152	3.58158	1.15230	2.48255	5.34348
1.54	2.3716	1.24097	3.92428	3.65226	1.15480	2.48794	5.36011
1.55	2.4025	1.24499	3.93700	3.72388	1.15729	2.49332	5.37169
1.56	2.4336	1.24900	3.94968	3.79642	1.15978	2.49867	5.38321
1.57	2.4649	1.25300	3.96232	3.86989	1.16225	2.50399	5.39469
1.58	2.4964	1.25698	3.97492	3.94431	1.16471	2.50930	5.40612
1.59	2.5281	1.26095	3.98748	4.01968	1.16717	2.51458	5.41750
1.60	2.5600	1.26491	4.00000	4.09600	1.16961	2.51984	5.42884
1.61	2.5921	1.26886	4.01248	4.17328	1.17204	2.52508	5.44012
1.62	2.6244	1.27279	4.02492	4.25153	1.17446	2.53030	5.45136
1.63	2.6569	1.27671	4.03733	4.33075	1.17687	2.53549	5.46256
1.64	2.6896	1.28062	4.04969	4.41094	1.17927	2.54067	5.47370
1.65	2.7225	1.28452	4.06202	4.49212	1.18167	2.54582	5.48481
1.66	2.7556	1.28841	4.07431	4.57430	1.18405	2.55095	5.49586
1.67	2.7889	1.29228	4.08656	4.65746	1.18642	2.55607	5.50688
1.68	2.8224	1.29615	4.09878	4.74163	1.18878	2.56116	5.51785
1.69	2.8561	1.30000	4.11096	4.82681	1.19114	2.56623	5.52877
1.70	2.8900	1.30384	4.12311	4.91300	1.19348	2.57128	5.53966
1.71	2.9241	1.30767	4.13521	5.00021	1.19582	2.57631	5.55050
1.72	2.9584	1.31149	4.14729	5.08845	1.19815	2.58133	5.56130
1.73	2.9929	1.31529	4.15933	5.17772	1.20046	2.58632	5.57205
1.74	3.0276	1.31909	4.17133	5.26802	1.20277	2.59129	5.58277
1.75	3.0625	1.32288	4.18330	5.35938	1.20507	2.59625	5.59344
1.76	3.0976	1.32665	4.19524	5.45178	1.20736	2.60118	5.60408
1.77	3.1329	1.33041	4.20714	5.54523	1.20964	2.60610	5.61467
1.78	3.1684	1.33417	4.21900	5.63975	1.21192	2.61100	5.62523
1.79	3.2041	1.33791	4.23084	5.73534	1.21418	2.61588	5.63574
1.80	3.2400	1.34164	4.24264	5.83200	1.21644	2.62074	5.64622
1.81	3.2761	1.34536	4.25441	5.92974	1.21869	2.62559	5.65665
1.82	3.3124	1.34907	4.26615	6.02857	1.22093	2.63041	5.66705
1.83	3.3489	1.35277	4.27785	6.12849	1.22316	2.63522	5.67741
1.84	3.3856	1.35647	4.28952	6.22950	1.22539	2.64001	5.68773
1.85	3.4225	1.36015	4.30116	6.33162	1.22760	2.64479	5.69802
1.86	3.4596	1.36382	4.31277	6.43486	1.22981	2.64954	5.70827
1.87	3.4969	1.36748	4.32435	6.53920	1.23201	2.65428	5.71848
1.88	3.5344	1.37113	4.33590	6.64467	1.23420	2.65901	5.72865
1.89	3.5721	1.37477	4.34741	6.75127	1.23639	2.66371	5.73879
1.90	3.6100	1.37840	4.35890	6.85900	1.23856	2.66840	5.74890
1.91	3.6481	1.38203	4.37035	6.96787	1.24073	2.67307	5.75897
1.92	3.6864	1.38564	4.38178	7.07789	1.24289	2.67773	5.76900
1.93	3.7249	1.38924	4.39318	7.18906	1.24505	2.68237	5.77900
1.94	3.7636	1.39284	4.40454	7.30138	1.24719	2.68700	5.78896
1.95	3.8025	1.39642	4.41588	7.41488	1.24933	2.69161	5.79889
1.96	3.8416	1.40000	4.42719	7.52954	1.25146	2.69620	5.80879
1.97	3.8809	1.40357	4.43847	7.64537	1.25359	2.70078	5.81865
1.98	3.9204	1.40712	4.44972	7.76239	1.25571	2.70534	5.82848
1.99	3.9601	1.41067	4.46094	7.88060	1.25782	2.70989	5.83827

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.00	4.0000	1.41421	4.47214	8.00000	1.25992	2.71442	5.84804
2.01	4.0401	1.41774	4.48330	8.12060	1.26202	2.71893	5.85777
2.02	4.0804	1.42127	4.49444	8.24241	1.26411	2.72344	5.86746
2.03	4.1209	1.42478	4.50555	8.36543	1.26619	2.72792	5.87713
2.04	4.1616	1.42829	4.51664	8.48966	1.26827	2.73239	5.88677
2.05	4.2025	1.43178	4.52769	8.61512	1.27033	2.73685	5.89637
2.06	4.2436	1.43527	4.53872	8.74182	1.27240	2.74129	5.90594
2.07	4.2849	1.43875	4.54973	8.86974	1.27445	2.74572	5.91548
2.08	4.3264	1.44222	4.56070	8.99891	1.27650	2.75014	5.92499
2.09	4.3681	1.44568	4.57165	9.12933	1.27854	2.75454	5.93447
2.10	4.4100	1.44914	4.58258	9.26100	1.28058	2.75892	5.94392
2.11	4.4521	1.45258	4.59347	9.39393	1.28261	2.76330	5.95334
2.12	4.4944	1.45602	4.60435	9.52813	1.28463	2.76766	5.96273
2.13	4.5369	1.45945	4.61519	9.66360	1.28665	2.77200	5.97209
2.14	4.5796	1.46287	4.62601	9.80034	1.28866	2.77633	5.98142
2.15	4.6225	1.46629	4.63681	9.93838	1.29066	2.78065	5.99073
2.16	4.6656	1.46969	4.64758	10.0777	1.29266	2.78495	6.00000
2.17	4.7089	1.47309	4.65833	10.2183	1.29465	2.78924	6.00925
2.18	4.7524	1.47648	4.66905	10.3602	1.29664	2.79352	6.01846
2.19	4.7961	1.47986	4.67974	10.5035	1.29862	2.79779	6.02765
2.20	4.8400	1.48324	4.69042	10.6480	1.30059	2.80204	6.03681
2.21	4.8841	1.48661	4.70106	10.7939	1.30256	2.80628	6.04594
2.22	4.9284	1.48997	4.71169	10.9410	1.30452	2.81050	6.05505
2.23	4.9729	1.49332	4.72229	11.0896	1.30648	2.81472	6.06413
2.24	5.0176	1.49666	4.73286	11.2394	1.30843	2.81892	6.07318
2.25	5.0625	1.50000	4.74342	11.3906	1.31037	2.82311	6.08220
2.26	5.1076	1.50333	4.75395	11.5432	1.31231	2.82728	6.09120
2.27	5.1529	1.50665	4.76445	11.6971	1.31424	2.83145	6.10017
2.28	5.1984	1.50997	4.77493	11.8524	1.31617	2.83560	6.10911
2.29	5.2441	1.51327	4.78539	12.0090	1.31809	2.83974	6.11803
2.30	5.2900	1.51658	4.79583	12.1670	1.32001	2.84387	6.12693
2.31	5.3361	1.51987	4.80625	12.3264	1.32192	2.84798	6.13579
2.32	5.3824	1.52315	4.81664	12.4872	1.32382	2.85209	6.14463
2.33	5.4289	1.52643	4.82701	12.6493	1.32572	2.85618	6.15345
2.34	5.4756	1.52971	4.83735	12.8129	1.32761	2.86026	6.16224
2.35	5.5225	1.53297	4.84768	12.9779	1.32950	2.86433	6.17101
2.36	5.5696	1.53623	4.85798	13.1443	1.33139	2.86838	6.17975
2.37	5.6169	1.53948	4.86826	13.3121	1.33326	2.87243	6.18846
2.38	5.6644	1.54272	4.87852	13.4813	1.33514	2.87646	6.19715
2.39	5.7121	1.54596	4.88876	13.6519	1.33700	2.88049	6.20582
2.40	5.7600	1.54919	4.89898	13.8240	1.33887	2.88450	6.21447
2.41	5.8081	1.55242	4.90918	13.9975	1.34072	2.88850	6.22308
2.42	5.8564	1.55563	4.91935	14.1725	1.34257	2.89249	6.23168
2.43	5.9049	1.55885	4.92950	14.3489	1.34442	2.89647	6.24025
2.44	5.9536	1.56205	4.93964	14.5268	1.34626	2.90044	6.24880
2.45	6.0025	1.56525	4.94975	14.7061	1.34810	2.90439	6.25732
2.46	6.0516	1.56844	4.95984	14.8869	1.34993	2.90834	6.26583
2.47	6.1009	1.57162	4.96991	15.0692	1.35176	2.91227	6.27431
2.48	6.1504	1.57480	4.97996	15.2530	1.35358	2.91620	6.28276
2.49	6.2001	1.57797	4.98999	15.4382	1.35540	2.92011	6.29119

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.50	6.2500	1.58114	5.00000	15.6250	1.35721	2.92402	6.29961
2.51	6.3001	1.58430	5.00999	15.8133	1.35902	2.92791	6.30799
2.52	6.3504	1.58745	5.01996	16.0030	1.36082	2.93179	6.31636
2.53	6.4009	1.59060	5.02991	16.1943	1.36262	2.93567	6.32470
2.54	6.4516	1.59374	5.03984	16.3871	1.36441	2.93953	6.33303
2.55	6.5025	1.59687	5.04975	16.5814	1.36620	2.94338	6.34133
2.56	6.5536	1.60000	5.05964	16.7772	1.36798	2.94723	6.34960
2.57	6.6049	1.60312	5.06952	16.9746	1.36976	2.95106	6.35786
2.58	6.6564	1.60624	5.07937	17.1735	1.37153	2.95488	6.36610
2.59	6.7081	1.60935	5.08920	17.3740	1.37330	2.95869	6.37431
2.60	6.7600	1.61245	5.09902	17.5760	1.37507	2.96250	6.38250
2.61	6.8121	1.61555	5.10882	17.7796	1.37683	2.96629	6.39068
2.62	6.8644	1.61864	5.11859	17.9847	1.37859	2.97007	6.39883
2.63	6.9169	1.62173	5.12835	18.1914	1.38034	2.97385	6.40696
2.64	6.9696	1.62481	5.13809	18.3997	1.38208	2.97761	6.41507
2.65	7.0225	1.62788	5.14782	18.6096	1.38383	2.98137	6.42316
2.66	7.0756	1.63095	5.15752	18.8211	1.38557	2.98511	6.43123
2.67	7.1289	1.63401	5.16720	19.0342	1.38730	2.98885	6.43928
2.68	7.1824	1.63707	5.17687	19.2488	1.38903	2.99257	6.44731
2.69	7.2361	1.64012	5.18652	19.4651	1.39076	2.99629	6.45531
2.70	7.2900	1.64317	5.19615	19.6830	1.39248	3.00000	6.46330
2.71	7.3441	1.64621	5.20577	19.9025	1.39419	3.00370	6.47127
2.72	7.3984	1.64924	5.21536	20.1236	1.39591	3.00739	6.47922
2.73	7.4529	1.65227	5.22494	20.3464	1.39761	3.01107	6.48715
2.74	7.5076	1.65529	5.23450	20.5708	1.39932	3.01474	6.49507
2.75	7.5625	1.65831	5.24404	20.7969	1.40102	3.01841	6.50296
2.76	7.6176	1.66132	5.25357	21.0246	1.40272	3.02206	6.51083
2.77	7.6729	1.66433	5.26308	21.2539	1.40441	3.02570	6.51868
2.78	7.7284	1.66733	5.27257	21.4850	1.40610	3.02934	6.52652
2.79	7.7841	1.67033	5.28205	21.7176	1.40778	3.03297	6.53434
2.80	7.8400	1.67332	5.29150	21.9520	1.40946	3.03659	6.54213
2.81	7.8961	1.67631	5.30094	22.1880	1.41114	3.04020	6.54991
2.82	7.9524	1.67929	5.31037	22.4258	1.41281	3.04380	6.55767
2.83	8.0089	1.68226	5.31977	22.6652	1.41448	3.04740	6.56541
2.84	8.0656	1.68523	5.32917	22.9063	1.41614	3.05098	6.57314
2.85	8.1225	1.68819	5.33854	23.1491	1.41780	3.05456	6.58084
2.86	8.1796	1.69115	5.34790	23.3937	1.41946	3.05813	6.58853
2.87	8.2369	1.69411	5.35724	23.6399	1.42111	3.06169	6.59620
2.88	8.2944	1.69706	5.36656	23.8879	1.42276	3.06524	6.60385
2.89	8.3521	1.70000	5.37587	24.1376	1.42440	3.06878	6.61149
2.90	8.4100	1.70294	5.38516	24.3890	1.42604	3.07232	6.61911
2.91	8.4681	1.70587	5.39444	24.6422	1.42768	3.07584	6.62671
2.92	8.5264	1.70880	5.40370	24.8971	1.42931	3.07936	6.63429
2.93	8.5849	1.71172	5.41295	25.1538	1.43094	3.08287	6.64185
2.94	8.6436	1.71464	5.42218	25.4122	1.43257	3.08638	6.64940
2.95	8.7025	1.71756	5.43139	25.6724	1.43419	3.08987	6.65693
2.96	8.7616	1.72047	5.44059	25.9343	1.43581	3.09336	6.66444
2.97	8.8209	1.72337	5.44977	26.1981	1.43743	3.09684	6.67194
2.98	8.8804	1.72627	5.45894	26.4636	1.43904	3.10031	6.67942
2.99	8.9401	1.72916	5.46809	26.7309	1.44065	3.10378	6.68688

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.00	9.0000	1.73205	5.47723	27.0000	1.44225	3.10723	6.69433
3.01	9.0601	1.73494	5.48635	27.2709	1.44385	3.11068	6.70176
3.02	9.1204	1.73781	5.49545	27.5436	1.44545	3.11412	6.70917
3.03	9.1809	1.74069	5.50454	27.8181	1.44704	3.11756	6.71657
3.04	9.2416	1.74356	5.51362	28.0945	1.44863	3.12098	6.72395
3.05	9.3025	1.74642	5.52268	28.3726	1.45022	3.12440	6.73132
3.06	9.3636	1.74929	5.53173	28.6526	1.45180	3.12781	6.73866
3.07	9.4249	1.75214	5.54076	28.9344	1.45338	3.13121	6.74600
3.08	9.4864	1.75499	5.54977	29.2181	1.45496	3.13461	6.75331
3.09	9.5481	1.75784	5.55878	29.5036	1.45653	3.13800	6.76061
3.10	9.6100	1.76068	5.56776	29.7910	1.45810	3.14138	6.76790
3.11	9.6721	1.76352	5.57674	30.0802	1.45967	3.14475	6.77517
3.12	9.7344	1.76635	5.58570	30.3713	1.46123	3.14812	6.78242
3.13	9.7969	1.76918	5.59464	30.6643	1.46279	3.15148	6.78966
3.14	9.8596	1.77200	5.60357	30.9591	1.46434	3.15483	6.79688
3.15	9.9225	1.77482	5.61249	31.2559	1.46590	3.15818	6.80409
3.16	9.9856	1.77764	5.62139	31.5545	1.46745	3.16152	6.81128
3.17	10.0489	1.78045	5.63028	31.8550	1.46899	3.16485	6.81846
3.18	10.1124	1.78326	5.63915	32.1574	1.47054	3.16817	6.82562
3.19	10.1761	1.78606	5.64801	32.4618	1.47208	3.17149	6.83277
3.20	10.2400	1.78885	5.65685	32.7680	1.47361	3.17480	6.83990
3.21	10.3041	1.79165	5.66569	33.0762	1.47515	3.17811	6.84702
3.22	10.3684	1.79444	5.67450	33.3862	1.47668	3.18140	6.85412
3.23	10.4329	1.79722	5.68331	33.6983	1.47820	3.18469	6.86121
3.24	10.4976	1.80000	5.69210	34.0122	1.47973	3.18798	6.86829
3.25	10.5625	1.80278	5.70088	34.3281	1.48125	3.19125	6.87534
3.26	10.6276	1.80555	5.70964	34.6460	1.48277	3.19452	6.88239
3.27	10.6929	1.80831	5.71839	34.9658	1.48428	3.19778	6.88942
3.28	10.7584	1.81108	5.72713	35.2876	1.48579	3.20104	6.89643
3.29	10.8241	1.81384	5.73585	35.6113	1.48730	3.20429	6.90344
3.30	10.8900	1.81659	5.74456	35.9370	1.48881	3.20753	6.91042
3.31	10.9561	1.81934	5.75326	36.2647	1.49031	3.21077	6.91740
3.32	11.0224	1.82209	5.76194	36.5944	1.49181	3.21400	6.92436
3.33	11.0889	1.82483	5.77062	36.9260	1.49330	3.21722	6.93130
3.34	11.1556	1.82757	5.77927	37.2597	1.49480	3.22044	6.93823
3.35	11.2225	1.83030	5.78792	37.5954	1.49629	3.22365	6.94515
3.36	11.2896	1.83303	5.79655	37.9331	1.49777	3.22686	6.95205
3.37	11.3569	1.83576	5.80517	38.2728	1.49926	3.23006	6.95894
3.38	11.4244	1.83848	5.81378	38.6145	1.50074	3.23325	6.96582
3.39	11.4921	1.84120	5.82237	38.9582	1.50222	3.23643	6.97268
3.40	11.5600	1.84391	5.83095	39.3040	1.50369	3.23961	6.97953
3.41	11.6281	1.84662	5.83952	39.6518	1.50517	3.24278	6.98637
3.42	11.6964	1.84932	5.84808	40.0017	1.50664	3.24595	6.99319
3.43	11.7649	1.85203	5.85662	40.3536	1.50810	3.24911	7.00000
3.44	11.8336	1.85472	5.86515	40.7076	1.50957	3.25227	7.00680
3.45	11.9025	1.85742	5.87367	41.0636	1.51103	3.25542	7.01358
3.46	11.9716	1.86011	5.88218	41.4217	1.51249	3.25856	7.02035
3.47	12.0409	1.86279	5.89067	41.7819	1.51394	3.26169	7.02711
3.48	12.1104	1.86548	5.89915	42.1442	1.51540	3.26482	7.03385
3.49	12.1801	1.86815	5.90762	42.5085	1.51685	3.26795	7.04058

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.50	12.2500	1.87083	5.91608	42.8750	1.51829	3.27107	7.04730
3.51	12.3201	1.87350	5.92453	43.2436	1.51974	3.27418	7.05400
3.52	12.3904	1.87617	5.93296	43.6142	1.52118	3.27729	7.06070
3.53	12.4609	1.87883	5.94138	43.9870	1.52262	3.28039	7.06738
3.54	12.5316	1.88149	5.94979	44.3619	1.52406	3.28348	7.07404
3.55	12.6025	1.88414	5.95819	44.7389	1.52549	3.28657	7.08070
3.56	12.6736	1.88680	5.96657	45.1180	1.52692	3.28965	7.08734
3.57	12.7449	1.88944	5.97495	45.4993	1.52835	3.29273	7.09397
3.58	12.8164	1.89209	5.98331	45.8827	1.52978	3.29580	7.10059
3.59	12.8881	1.89473	5.99166	46.2683	1.53120	3.29887	7.10719
3.60	12.9600	1.89737	6.00000	46.6560	1.53262	3.30193	7.11379
3.61	13.0321	1.90000	6.00833	47.0459	1.53404	3.30498	7.12037
3.62	13.1044	1.90263	6.01664	47.4379	1.53545	3.30803	7.12694
3.63	13.1769	1.90526	6.02495	47.8321	1.53686	3.31107	7.13349
3.64	13.2496	1.90788	6.03324	48.2285	1.53827	3.31411	7.14004
3.65	13.3225	1.91050	6.04152	48.6271	1.53968	3.31714	7.14657
3.66	13.3956	1.91311	6.04979	49.0279	1.54109	3.32017	7.15309
3.67	13.4689	1.91572	6.05805	49.4309	1.54249	3.32319	7.15960
3.68	13.5424	1.91833	6.06630	49.8360	1.54389	3.32621	7.16610
3.69	13.6161	1.92094	6.07454	50.2434	1.54529	3.32922	7.17258
3.70	13.6900	1.92354	6.08276	50.6530	1.54668	3.33222	7.17905
3.71	13.7641	1.92614	6.09098	51.0648	1.54807	3.33522	7.18552
3.72	13.8384	1.92873	6.09918	51.4788	1.54946	3.33822	7.19197
3.73	13.9129	1.93132	6.10737	51.8951	1.55085	3.34120	7.19840
3.74	13.9876	1.93391	6.11555	52.3136	1.55223	3.34419	7.20483
3.75	14.0625	1.93649	6.12372	52.7344	1.55362	3.34716	7.21125
3.76	14.1376	1.93907	6.13188	53.1574	1.55500	3.35014	7.21765
3.77	14.2129	1.94165	6.14003	53.5826	1.55637	3.35310	7.22405
3.78	14.2884	1.94422	6.14817	54.0102	1.55775	3.35607	7.23043
3.79	14.3641	1.94679	6.15630	54.4399	1.55912	3.35902	7.23680
3.80	14.4400	1.94936	6.16441	54.8720	1.56049	3.36198	7.24316
3.81	14.5161	1.95192	6.17252	55.3063	1.56186	3.36492	7.24950
3.82	14.5924	1.95448	6.18061	55.7430	1.56322	3.36786	7.25584
3.83	14.6689	1.95704	6.18870	56.1819	1.56459	3.37080	7.26217
3.84	14.7456	1.95959	6.19677	56.6231	1.56595	3.37373	7.26848
3.85	14.8225	1.96214	6.20484	57.0666	1.56731	3.37666	7.27479
3.86	14.8996	1.96469	6.21289	57.5125	1.56866	3.37958	7.28108
3.87	14.9769	1.96723	6.22093	57.9606	1.57001	3.38249	7.28736
3.88	15.0544	1.96977	6.22896	58.4111	1.57137	3.38540	7.29363
3.89	15.1321	1.97231	6.23699	58.8639	1.57271	3.38831	7.29989
3.90	15.2100	1.97484	6.24500	59.3190	1.57406	3.39121	7.30614
3.91	15.2881	1.97737	6.25300	59.7765	1.57541	3.39411	7.31238
3.92	15.3664	1.97990	6.26099	60.2363	1.57675	3.39700	7.31861
3.93	15.4449	1.98242	6.26897	60.6985	1.57809	3.39988	7.32483
3.94	15.5236	1.98494	6.27694	61.1630	1.57942	3.40277	7.33104
3.95	15.6025	1.98746	6.28490	61.6299	1.58076	3.40564	7.33723
3.96	15.6816	1.98997	6.29285	62.0991	1.58209	3.40851	7.34342
3.97	15.7609	1.99249	6.30079	62.5708	1.58342	3.41138	7.34960
3.98	15.8404	1.99499	6.30872	63.0448	1.58475	3.41424	7.35576
3.99	15.9201	1.99750	6.31664	63.5212	1.58608	3.41710	7.36192

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.00	16.0000	2.00000	6.32456	64.0000	1.58740	3.41995	7.36806
4.01	16.0801	2.00250	6.33246	64.4812	1.58872	3.42280	7.37420
4.02	16.1604	2.00499	6.34035	64.9648	1.59004	3.42564	7.38032
4.03	16.2409	2.00749	6.34823	65.4508	1.59136	3.42848	7.38644
4.04	16.3216	2.00998	6.35610	65.9393	1.59267	3.43131	7.39254
4.05	16.4025	2.01246	6.36396	66.4301	1.59399	3.43414	7.39864
4.06	16.4836	2.01494	6.[illegible]181	66.9234	1.59530	3.43697	7.40472
4.07	16.5649	2.01742	6.37966	67.4191	1.59661	3.43979	7.41080
4.08	16.6464	2.01990	6.38749	67.9173	1.59791	3.44260	7.41686
4.09	16.7281	2.02237	6.39531	68.4179	1.59922	3.44541	7.42291
4.10	16.8100	2.02485	6.40312	68.9210	1.60052	3.44822	7.42896
4.11	16.8921	2.02731	6.41093	69.4265	1.60182	3.45102	7.43499
4.12	16.9744	2.02978	6.41872	69.9345	1.60312	3.45382	7.44102
4.13	17.0569	2.03224	6.42651	70.4450	1.60441	3.45661	7.44703
4.14	17.1396	2.03470	6.43428	70.9579	1.60571	3.45939	7.45304
4.15	17.2225	2.03715	6.44205	71.4734	1.60700	3.46218	7.45904
4.16	17.3056	2.03961	6.44981	71.9913	1.60829	3.46496	7.46502
4.17	17.3889	2.04206	6.45755	72.5117	1.60958	3.46773	7.47100
4.18	17.4724	2.04450	6.46529	73.0346	1.61086	3.47050	7.47697
4.19	17.5561	2.04695	6.47302	73.5601	1.61215	3.47327	7.48292
4.20	17.6400	2.04939	6.48074	74.0880	1.61343	3.47603	7.48887
4.21	17.7241	2.05183	6.48845	74.6185	1.61471	3.47878	7.49481
4.22	17.8084	2.05426	6.49615	75.1514	1.61599	3.48154	7.50074
4.23	17.8929	2.05670	6.50384	75.6870	1.61726	3.48428	7.50666
4.24	17.9776	2.05913	6.51153	76.2250	1.61853	3.48703	7.51257
4.25	18.0625	2.06155	6.51920	76.7656	1.61981	3.48977	7.51847
4.26	18.1476	2.06398	6.52687	77.3088	1.62108	3.49250	7.52437
4.27	18.2329	2.06640	6.53452	77.8545	1.62234	3.49523	7.53025
4.28	18.3184	2.06882	6.54217	78.4028	1.62361	3.49796	7.53612
4.29	18.4041	2.07123	6.54981	78.9536	1.62487	3.50068	7.54199
4.30	18.4900	2.07364	6.55744	79.5070	1.62613	3.50340	7.54784
4.31	18.5761	2.07605	6.56506	80.0630	1.62739	3.50611	7.55369
4.32	18.6624	2.07846	6.57267	80.6216	1.62865	3.50882	7.55953
4.33	18.7489	2.08087	6.58027	81.1827	1.62991	3.51153	7.56535
4.34	18.8356	2.08327	6.58787	81.7465	1.63116	3.51423	7.57117
4.35	18.9225	2.08567	6.59545	82.3129	1.63241	3.51692	7.57698
4.36	19.0096	2.08806	6.60303	82.8819	1.63366	3.51962	7.58279
4.37	19.0969	2.09045	6.61060	83.4535	1.63491	3.52231	7.58858
4.38	19.1844	2.09284	6.61816	84.0277	1.63619	3.52499	7.59436
4.39	19.2721	2.09523	6.62571	84.6045	1.63740	3.52767	7.60014
4.40	19.3600	2.09762	6.63325	85.1840	1.63864	3.53035	7.60590
4.41	19.4481	2.10000	6.64078	85.7661	1.63988	3.53302	7.61166
4.42	19.5364	2.10238	6.64831	86.3509	1.64112	3.53569	7.61741
4.43	19.6249	2.10476	6.65582	86.9383	1.64236	3.53835	7.62315
4.44	19.7136	2.10713	6.66333	87.5284	1.64359	3.54101	7.62888
4.45	19.8025	2.10950	6.67083	88.1211	1.64483	3.54367	7.63461
4.46	19.8916	2.11187	6.67832	88.7165	1.64606	3.54632	7.64032
4.47	19.9809	2.11424	6.68581	89.3146	1.64729	3.54897	7.64603
4.48	20.0704	2.11660	6.69328	89.9154	1.64851	3.55162	7.65172
4.49	20.1601	2.11896	6.70075	90.5188	1.64974	3.55426	7.65741

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.50	20.2500	2.12132	6.70820	91.1250	1.65096	3.55689	7.66309
4.51	20.3401	2.12368	6.71565	91.7339	1.65219	3.55953	7.66877
4.52	20.4304	2.12603	6.72309	92.3454	1.65341	3.56215	7.67443
4.53	20.5209	2.12838	6.73053	92.9597	1.65462	3.56478	7.68009
4.54	20.6116	2.13073	6.73795	93.5767	1.65584	3.56740	7.68573
4.55	20.7025	2.13307	6.74537	94.1964	1.65706	3.57002	7.69137
4.56	20.7936	2.13542	6.75278	94.8188	1.65827	3.57263	7.69700
4.57	20.8849	2.13776	6.76018	95.4440	1.65948	3.57524	7.70262
4.58	20.9764	2.14009	6.76757	96.0719	1.66069	3.57785	7.70824
4.59	21.0681	2.14243	6.77495	96.7026	1.66190	3.58045	7.71384
4.60	21.1600	2.14476	6.78233	97.3360	1.66310	3.58305	7.71944
4.61	21.2521	2.14709	6.78970	97.9722	1.66431	3.58564	7.72503
4.62	21.3444	2.14942	6.79706	98.6111	1.66551	3.58823	7.73061
4.63	21.4369	2.15174	6.80441	99.2528	1.66671	3.59082	7.73619
4.64	21.5296	2.15407	6.81175	99.8973	1.66791	3.59340	7.74175
4.65	21.6225	2.15639	6.81909	100.545	1.66911	3.59598	7.74731
4.66	21.7156	2.15870	6.82642	101.195	1.67030	3.59856	7.75286
4.67	21.8089	2.16102	6.83374	101.848	1.67150	3.60113	7.75840
4.68	21.9024	2.16333	6.84105	102.503	1.67269	3.60370	7.76394
4.69	21.9961	2.16564	6.84836	103.162	1.67388	3.60626	7.76946
4.70	22.0900	2.16795	6.85565	103.823	1.67507	3.60883	7.77498
4.71	22.1841	2.17025	6.86294	104.487	1.67626	3.61138	7.78049
4.72	22.2784	2.17256	6.87023	105.154	1.67744	3.61394	7.78599
4.73	22.3729	2.17486	6.87750	105.824	1.67863	3.61649	7.79149
4.74	22.4676	2.17715	6.88477	106.496	1.67981	3.61903	7.79697
4.75	22.5625	2.17945	6.89202	107.172	1.68099	3.62158	7.80245
4.76	22.6576	2.18174	6.89928	107.850	1.68217	3.62412	7.80793
4.77	22.7529	2.18403	6.90652	108.531	1.68334	3.62665	7.81339
4.78	22.8484	2.18632	6.91375	109.215	1.68452	3.62919	7.81885
4.79	22.9441	2.18861	6.92098	109.902	1.68569	3.63172	7.82429
4.80	23.0400	2.19089	6.92820	110.592	1.68687	3.63424	7.82974
4.81	23.1361	2.19317	6.93542	111.285	1.68804	3.63676	7.83517
4.82	23.2324	2.19545	6.94262	111.980	1.68920	3.63928	7.84059
4.83	23.3289	2.19773	6.94982	112.679	1.69037	3.64180	7.84601
4.84	23.4256	2.20000	6.95701	113.380	1.69154	3.64431	7.85142
4.85	23.5225	2.20227	6.96419	114.084	1.69270	3.64682	7.85683
4.86	23.6196	2.20454	6.97137	114.791	1.69386	3.64932	7.86222
4.87	23.7169	2.20681	6.97854	115.501	1.69503	3.65182	7.86761
4.88	23.8144	2.20907	6.98570	116.214	1.69619	3.65432	7.87299
4.89	23.9121	2.21133	6.99285	116.930	1.69734	3.65681	7.87837
4.90	24.0100	2.21359	7.00000	117.649	1.69850	3.65931	7.88374
4.91	24.1081	2.21585	7.00714	118.371	1.69965	3.66179	7.88909
4.92	24.2064	2.21811	7.01427	119.095	1.70081	3.66428	7.89445
4.93	24.3049	2.22036	7.02140	119.823	1.70196	3.66676	7.89979
4.94	24.4036	2.22261	7.02851	120.554	1.70311	3.66924	7.90513
4.95	24.5025	2.22486	7.03562	121.287	1.70426	3.67171	7.91046
4.96	24.6016	2.22711	7.04273	122.024	1.70540	3.67418	7.91578
4.97	24.7009	2.22935	7.04982	122.763	1.70655	3.67665	7.92110
4.98	24.8004	2.23159	7.05691	123.506	1.70769	3.67911	7.92641
4.99	24.9001	2.23383	7.06399	124.251	1.70884	3.68157	7.93171

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.00	25.0000	2.23607	7.07107	125.000	1.70998	3.68403	7.93701
5.01	25.1001	2.23830	7.07814	125.752	1.71112	3.68649	7.94229
5.02	25.2004	2.24054	7.08520	126.506	1.71225	3.68894	7.94757
5.03	25.3009	2.24277	7.09225	127.264	1.71339	3.69138	7.95285
5.04	25.4016	2.24499	7.09930	128.024	1.71452	3.69383	7.95811
5.05	25.5025	2.24722	7.10634	128.788	1.71566	3.69627	7.96337
5.06	25.6036	2.24944	7.11337	129.554	1.71679	3.69871	7.96863
5.07	25.7049	2.25167	7.12039	130.324	1.71792	3.70114	7.97387
5.08	25.8064	2.25389	7.12741	131.097	1.71905	3.70357	7.97911
5.09	25.9081	2.25610	7.13442	131.872	1.72017	3.70600	7.98434
5.10	26.0100	2.25832	7.14143	132.651	1.72130	3.70843	7.98957
5.11	26.1121	2.26053	7.14843	133.433	1.72242	3.71085	7.99479
5.12	26.2144	2.26274	7.15542	134.218	1.72355	3.71327	8.00000
5.13	26.3169	2.26495	7.16240	135.006	1.72467	3.71569	8.00520
5.14	26.4196	2.26716	7.16938	135.797	1.72579	3.71810	8.01040
5.15	26.5225	2.26936	7.17635	136.591	1.72691	3.72051	8.01559
5.16	26.6256	2.27156	7.18331	137.388	1.72802	3.72292	8.02078
5.17	26.7289	2.27376	7.19027	138.188	1.72914	3.72532	8.02596
5.18	26.8324	2.27596	7.19722	138.992	1.73025	3.72772	8.03113
5.19	26.9361	2.27816	7.20417	139.798	1.73137	3.73012	8.03629
5.20	27.0400	2.28035	7.21110	140.608	1.73248	3.73251	8.04145
5.21	27.1441	2.28254	7.21803	141.421	1.73359	3.73490	8.04660
5.22	27.2484	2.28473	7.22496	142.237	1.73470	3.73729	8.05175
5.23	27.3529	2.28692	7.23187	143.056	1.73580	3.73968	8.05689
5.24	27.4576	2.28910	7.23878	143.878	1.73691	3.74206	8.06202
5.25	27.5625	2.29129	7.24569	144.703	1.73801	3.74443	8.06714
5.26	27.6676	2.29347	7.25259	145.532	1.73912	3.74681	8.07226
5.27	27.7729	2.29565	7.25948	146.363	1.74022	3.74918	8.07737
5.28	27.8784	2.29783	7.26636	147.198	1.74132	3.75155	8.08248
5.29	27.9841	2.30000	7.27324	148.036	1.74242	3.75392	8.08758
5.30	28.0900	2.30217	7.28011	148.877	1.74351	3.75629	8.09267
5.31	28.1961	2.30434	7.28697	149.721	1.74461	3.75865	8.09776
5.32	28.3024	2.30651	7.29383	150.569	1.74570	3.76101	8.10284
5.33	28.4089	2.30868	7.30068	151.419	1.74680	3.76336	8.10791
5.34	28.5156	2.31084	7.30753	152.273	1.74789	3.76571	8.11298
5.35	28.6225	2.31301	7.31437	153.130	1.74898	3.76806	8.11804
5.36	28.7296	2.31517	7.32120	153.991	1.75007	3.77041	8.12310
5.37	28.8369	2.31733	7.32803	154.854	1.75116	3.77275	8.12814
5.38	28.9444	2.31948	7.33485	155.721	1.75224	3.77509	8.13319
5.39	29.0521	2.32164	7.34166	156.591	1.75333	3.77743	8.13822
5.40	29.1600	2.32379	7.34847	157.464	1.75441	3.77976	8.14325
5.41	29.2681	2.32594	7.35527	158.340	1.75549	3.78209	8.14828
5.42	29.3764	2.32809	7.36206	159.220	1.75657	3.78442	8.15329
5.43	29.4849	2.33024	7.36885	160.103	1.75765	3.78675	8.15831
5.44	29.5936	2.33238	7.37564	160.989	1.75873	3.78907	8.16331
5.45	29.7025	2.33452	7.38241	161.879	1.75981	3.79139	8.16831
5.46	29.8116	2.33666	7.38918	162.771	1.76088	3.79371	8.17330
5.47	29.9209	2.33880	7.39594	163.667	1.76196	3.79603	8.17829
5.48	30.0304	2.34094	7.40270	164.567	1.76303	3.79834	8.18327
5.49	30.1401	2.34307	7.40945	165.469	1.76410	3.80065	8.18824

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.50	30.2500	2.34521	7.41620	166.375	1.76517	3.80295	8.19321
5.51	30.3601	2.34734	7.42294	167.284	1.76624	3.80526	8.19818
5.52	30.4704	2.34947	7.42967	168.197	1.76731	3.80756	8.20313
5.53	30.5809	2.35160	7.43640	169.112	1.76838	3.80985	8.20808
5.54	30.6916	2.35372	7.44312	170.031	1.76944	3.81215	8.21303
5.55	30.8025	2.35584	7.44983	170.954	1.77051	3.81444	8.21797
5.56	30.9136	2.35797	7.45654	171.880	1.77157	3.81673	8.22290
5.57	31.0249	2.36008	7.46324	172.809	1.77263	3.81902	8.22783
5.58	31.1364	2.36220	7.46994	173.741	1.77369	3.82130	8.23275
5.59	31.2481	2.36432	7.47663	174.677	1.77475	3.82358	8.23766
5.60	31.3600	2.36643	7.48331	175.616	1.77581	3.82586	8.24257
5.61	31.4721	2.36854	7.48999	176.558	1.77686	3.82814	8.24747
5.62	31.5844	2.37065	7.49667	177.504	1.77792	3.83041	8.25237
5.63	31.6969	2.37276	7.50333	178.454	1.77897	3.83268	8.25726
5.64	31.8096	2.37487	7.50999	179.406	1.78003	3.83495	8.26215
5.65	31.9225	2.37697	7.51665	180.362	1.78108	3.83722	8.26703
5.66	32.0356	2.37908	7.52330	181.321	1.78213	3.83948	8.27190
5.67	32.1489	2.38118	7.52994	182.284	1.78318	3.84174	8.27677
5.68	32.2624	2.38328	7.53658	183.250	1.78422	3.84399	8.28164
5.69	32.3761	2.38537	7.54321	184.220	1.78527	3.84625	8.28649
5.70	32.4900	2.38747	7.54983	185.193	1.78632	3.84850	8.29134
5.71	32.6041	2.38956	7.55645	186.169	1.78736	3.85075	8.29619
5.72	32.7184	2.39165	7.56307	187.149	1.78840	3.85300	8.30103
5.73	32.8329	2.39374	7.56968	188.133	1.78944	3.85524	8.30587
5.74	32.9476	2.39583	7.57628	189.119	1.79048	3.85748	8.31069
5.75	33.0625	2.39792	7.58288	190.109	1.79152	3.85972	8.31552
5.76	33.1776	2.40000	7.58947	191.103	1.79256	3.86196	8.32034
5.77	33.2929	2.40208	7.59605	192.100	1.79360	3.86419	8.32515
5.78	33.4084	2.40416	7.60263	193.101	1.79463	3.86642	8.32995
5.79	33.5241	2.40624	7.60920	194.105	1.79567	3.86865	8.33476
5.80	33.6400	2.40832	7.61577	195.112	1.79670	3.87088	8.33955
5.81	33.7561	2.41039	7.62234	196.123	1.79773	3.87310	8.34434
5.82	33.8724	2.41247	7.62889	197.137	1.79876	3.87532	8.34913
5.83	33.9889	2.41454	7.63544	198.155	1.79979	3.87754	8.35390
5.84	34.1056	2.41661	7.64199	199.177	1.80082	3.87975	8.35868
5.85	34.2225	2.41868	7.64853	200.202	1.80185	3.88197	8.36345
5.86	34.3396	2.42074	7.65506	201.230	1.80288	3.88418	8.36821
5.87	34.4569	2.42281	7.66159	202.262	1.80390	3.88639	8.37297
5.88	34.5744	2.42487	7.66812	203.297	1.80492	3.88859	8.37772
5.89	34.6921	2.42693	7.67463	204.336	1.80595	3.89080	8.38247
5.90	34.8100	2.42899	7.68115	205.379	1.80697	3.89300	8.38721
5.91	34.9281	2.43105	7.68765	206.425	1.80799	3.89519	8.39194
5.92	35.0464	2.43311	7.69415	207.475	1.80901	3.89739	8.39667
5.93	35.1649	2.43516	7.70065	208.528	1.81003	3.89958	8.40140
5.94	35.2836	2.43721	7.70714	209.585	1.81104	3.90177	8.40612
5.95	35.4025	2.43926	7.71362	210.645	1.81206	3.90396	8.41083
5.96	35.5216	2.44131	7.72010	211.709	1.81307	3.90615	8.41554
5.97	35.6409	2.44336	7.72658	212.776	1.81409	3.90833	8.42025
5.98	35.7604	2.44540	7.73305	213.847	1.81510	3.91051	8.42494
5.99	35.8801	2.44745	7.73951	214.922	1.81611	3.91269	8.42964

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.00	36.0000	2.44949	7.74597	216.000	1.81712	3.91487	8.43433
6.01	36.1201	2.45153	7.75242	217.082	1.81813	3.91704	8.43901
6.02	36.2404	2.45357	7.75887	218.167	1.81914	3.91921	8.44369
6.03	36.3609	2.45561	7.76531	219.256	1.82014	3.92138	8.44836
6.04	36.4816	2.45764	7.77174	220.349	1.82115	3.92355	8.45303
6.05	36.6025	2.45967	7.77817	221.445	1.82215	3.92571	8.45769
6.06	36.7236	2.46171	7.78460	222.545	1.82316	3.92787	8.46235
6.07	36.8449	2.46374	7.79102	223.649	1.82416	3.93003	8.46700
6.08	36.9664	2.46577	7.79744	224.756	1.82516	3.93219	8.47165
6.09	37.0881	2.46779	7.80385	225.867	1.82616	3.93434	8.47629
6.10	37.2100	2.46982	7.81025	226.981	1.82716	3.93650	8.48093
6.11	37.3321	2.47184	7.81665	228.099	1.82816	3.93865	8.48556
6.12	37.4544	2.47386	7.82304	229.221	1.82915	3.94079	8.49018
6.13	37.5769	2.47588	7.82943	230.346	1.83015	3.94294	8.49481
6.14	37.6996	2.47790	7.83582	231.476	1.83115	3.94508	8.49942
6.15	37.8225	2.47992	7.84219	232.608	1.83214	3.94722	8.50403
6.16	37.9456	2.48193	7.84857	233.745	1.83313	3.94936	8.50864
6.17	38.0689	2.48395	7.85493	234.885	1.83412	3.95150	8.51324
6.18	38.1924	2.48596	7.86130	236.029	1.83511	3.95363	8.51784
6.19	38.3161	2.48797	7.86766	237.177	1.83610	3.95576	8.52243
6.20	38.4400	2.48998	7.87401	238.328	1.83709	3.95789	8.52702
6.21	38.5641	2.49199	7.88036	239.483	1.83808	3.96002	8.53160
6.22	38.6884	2.49399	7.88670	240.642	1.83906	3.96214	8.53618
6.23	38.8129	2.49600	7.89303	241.804	1.84005	3.96427	8.54075
6.24	38.9376	2.49800	7.89937	242.971	1.84103	3.96638	8.54532
6.25	39.0625	2.50000	7.90569	244.141	1.84202	3.96850	8.54988
6.26	39.1876	2.50200	7.91202	245.314	1.84300	3.97062	8.55444
6.27	39.3129	2.50400	7.91833	246.492	1.84398	3.97273	8.55899
6.28	39.4384	2.50599	7.92465	247.673	1.84496	3.97484	8.56354
6.29	39.5641	2.50799	7.93095	248.858	1.84594	3.97695	8.56808
6.30	39.6900	2.50998	7.93725	250.047	1.84691	3.97906	8.57262
6.31	39.8161	2.51197	7.94355	251.240	1.84789	3.98116	8.57715
6.32	39.9424	2.51396	7.94984	252.436	1.84887	3.98326	8.58168
6.33	40.0689	2.51595	7.95613	253.636	1.84984	3.98536	8.58620
6.34	40.1956	2.51794	7.96241	254.840	1.85082	3.98746	8.59072
6.35	40.3225	2.51992	7.96869	256.048	1.85179	3.98956	8.59524
6.36	40.4496	2.52190	7.97496	257.259	1.85276	3.99165	8.59975
6.37	40.5769	2.52389	7.98123	258.475	1.85373	3.99374	8.60425
6.38	40.7044	2.52587	7.98749	259.694	1.85470	3.99583	8.60875
6.39	40.8321	2.52784	7.99375	260.917	1.85567	3.99792	8.61325
6.40	40.9600	2.52982	8.00000	262.144	1.85664	4.00000	8.61774
6.41	41.0881	2.53180	8.00625	263.375	1.85760	4.00208	8.62222
6.42	41.2164	2.53377	8.01249	264.609	1.85857	4.00416	8.62671
6.43	41.3449	2.53574	8.01873	265.848	1.85953	4.00624	8.63118
6.44	41.4736	2.53772	8.02496	267.090	1.86050	4.00832	8.63566
6.45	41.6025	2.53969	8.03119	268.336	1.86146	4.01039	8.64012
6.46	41.7316	2.54165	8.03741	269.586	1.86242	4.01246	8.64459
6.47	41.8609	2.54362	8.04363	270.840	1.86338	4.01453	8.64904
6.48	41.9904	2.54558	8.04984	272.098	1.86434	4.01660	8.65350
6.49	42.1201	2.54755	8.05605	273.359	1.86530	4.01866	8.65795

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.50	42.2500	2.54951	8.06226	274.625	1.86626	4.02073	8.66239
6.51	42.3801	2.55147	8.06846	275.894	1.86721	4.02279	8.66683
6.52	42.5104	2.55343	8.07465	277.168	1.86817	4.02485	8.67127
6.53	42.6409	2.55539	8.08084	278.445	1.86912	4.02690	8.67570
6.54	42.7716	2.55734	8.08703	279.726	1.87008	4.02896	8.68012
6.55	42.9025	2.55930	8.09321	281.011	1.87103	4.03101	8.68455
6.56	43.0336	2.56125	8.09938	282.300	1.87198	4.03306	8.68896
6.57	43.1649	2.56320	8.10555	283.593	1.87293	4.03511	8.69338
6.58	43.2964	2.56515	8.11172	284.890	1.87388	4.03715	8.69778
6.59	43.4281	2.56710	8.11788	286.191	1.87483	4.03920	8.70219
6.60	43.5600	2.56905	8.12404	287.496	1.87578	4.04124	8.70659
6.61	43.6921	2.57099	8.13019	288.805	1.87672	4.04328	8.71098
6.62	43.8244	2.57294	8.13634	290.118	1.87767	4.04532	8.71537
6.63	43.9569	2.57488	8.14248	291.434	1.87862	4.04735	8.71976
6.64	44.0896	2.57682	8.14862	292.755	1.87956	4.04939	8.72414
6.65	44.2225	2.57876	8.15475	294.080	1.88050	4.05142	8.72852
6.66	44.3556	2.58070	8.16088	295.408	1.88144	4.05345	8.73289
6.67	44.4889	2.58263	8.16701	296.741	1.88239	4.05548	8.73726
6.68	44.6224	2.58457	8.17313	298.078	1.88333	4.05750	8.74162
6.69	44.7561	2.58650	8.17924	299.418	1.88427	4.05953	8.74598
6.70	44.8900	2.58844	8.18535	300.763	1.88520	4.06155	8.75034
6.71	45.0241	2.59037	8.19146	302.112	1.88614	4.06357	8.75469
6.72	45.1584	2.59230	8.19756	303.464	1.88708	4.06559	8.75904
6.73	45.2929	2.59422	8.20366	304.821	1.88801	4.06760	8.76338
6.74	45.4276	2.59615	8.20975	306.182	1.88895	4.06961	8.76772
6.75	45.5625	2.59808	8.21584	307.547	1.88988	4.07163	8.77205
6.76	45.6976	2.60000	8.22192	308.916	1.89081	4.07364	8.77638
6.77	45.8329	2.60192	8.22800	310.289	1.89175	4.07564	8.78071
6.78	45.9684	2.60384	8.23408	311.666	1.89268	4.07765	8.78503
6.79	46.1041	2.60576	8.24015	313.047	1.89361	4.07965	8.78935
6.80	46.2400	2.60768	8.24621	314.432	1.89454	4.08166	8.79366
6.81	46.3761	2.60960	8.25227	315.821	1.89546	4.08365	8.79797
6.82	46.5124	2.61151	8.25833	317.215	1.89639	4.08565	8.80227
6.83	46.6489	2.61343	8.26438	318.612	1.89732	4.08765	8.80657
6.84	46.7856	2.61534	8.27043	320.014	1.89824	4.08964	8.81087
6.85	46.9225	2.61725	8.27647	321.419	1.89917	4.09163	8.81516
6.86	47.0596	2.61916	8.28251	322.829	1.90009	4.09362	8.81945
6.87	47.1969	2.62107	8.28855	324.243	1.90102	4.09561	8.82373
6.88	47.3344	2.62298	8.29458	325.661	1.90194	4.09760	8.82801
6.89	47.4721	2.62488	8.30060	327.083	1.90286	4.09958	8.83228
6.90	47.6100	2.62679	8.30662	328.509	1.90378	4.10157	8.83656
6.91	47.7481	2.62869	8.31264	329.939	1.90470	4.10355	8.84082
6.92	47.8864	2.63059	8.31865	331.374	1.90562	4.10552	8.84509
6.93	48.0249	2.63249	8.32466	332.813	1.90653	4.10750	8.84934
6.94	48.1636	2.63439	8.33067	334.255	1.90745	4.10948	8.85360
6.95	48.3025	2.63629	8.33667	335.702	1.90837	4.11145	8.85785
6.96	48.4416	2.63818	8.34266	337.154	1.90928	4.11342	8.86210
6.97	48.5809	2.64008	8.34865	338.609	1.91019	4.11539	8.86634
6.98	48.7204	2.64197	8.35464	340.068	1.91111	4.11736	8.87058
6.99	48.8601	2.64386	8.36062	341.532	1.91202	4.11932	8.87481

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.00	49.0000	2.64575	8.36660	343.000	1.91293	4.12129	8.87904
7.01	49.1401	2.64764	8.37257	344.472	1.91384	4.12325	8.88327
7.02	49.2804	2.64953	8.37854	345.948	1.91475	4.12521	8.88749
7.03	49.4209	2.65141	8.38451	347.429	1.91566	4.12716	8.89171
7.04	49.5616	2.65330	8.39047	348.914	1.91657	4.12912	8.89592
7.05	49.7025	2.65518	8.39643	350.403	1.91747	4.13107	8.90013
7.06	49.8436	2.65707	8.40238	351.896	1.91838	4.13303	8.90434
7.07	49.9849	2.65895	8.40833	353.393	1.91929	4.13498	8.90854
7.08	50.1264	2.66083	8.41427	354.895	1.92019	4.13693	8.91274
7.09	50.2681	2.66271	8.42021	356.401	1.92109	4.13887	8.91693
7.10	50.4100	2.66458	8.42615	357.911	1.92200	4.14082	8.92112
7.11	50.5521	2.66646	8.43208	359.425	1.92290	4.14276	8.92531
7.12	50.6944	2.66833	8.43801	360.944	1.92380	4.14470	8.92949
7.13	50.8369	2.67021	8.44393	362.467	1.92470	4.14664	8.93367
7.14	50.9796	2.67208	8.44985	363.994	1.92560	4.14858	8.93784
7.15	51.1225	2.67395	8.45577	365.526	1.92650	4.15052	8.94201
7.16	51.2656	2.67582	8.46168	367.062	1.92740	4.15245	8.94618
7.17	51.4089	2.67769	8.46759	368.602	1.92829	4.15438	8.95034
7.18	51.5524	2.67955	8.47349	370.146	1.92919	4.15631	8.95450
7.19	51.6961	2.68142	8.47939	371.695	1.93008	4.15824	8.95866
7.20	51.8400	2.68328	8.48528	373.248	1.93098	4.16017	8.96281
7.21	51.9841	2.68514	8.49117	374.805	1.93187	4.16209	8.96696
7.22	52.1284	2.68701	8.49706	376.367	1.93277	4.16402	8.97110
7.23	52.2729	2.68887	8.50294	377.933	1.93366	4.16594	8.97524
7.24	52.4176	2.69072	8.50882	379.503	1.93455	4.16786	8.97938
7.25	52.5625	2.69258	8.51469	381.078	1.93544	4.16978	8.98351
7.26	52.7076	2.69444	8.52056	382.657	1.93633	4.17169	8.98764
7.27	52.8529	2.69629	8.52643	384.241	1.93722	4.17361	8.99176
7.28	52.9984	2.69815	8.53229	385.828	1.93810	4.17552	8.99588
7.29	53.1441	2.70000	8.53815	387.420	1.93899	4.17743	9.00000
7.30	53.2900	2.70185	8.54400	389.017	1.93988	4.17934	9.00411
7.31	53.4361	2.70370	8.54985	390.618	1.94076	4.18125	9.00822
7.32	53.5824	2.70555	8.55570	392.223	1.94165	4.18315	9.01233
7.33	53.7289	2.70740	8.56154	393.833	1.94253	4.18506	9.01643
7.34	53.8756	2.70924	8.56738	395.447	1.94341	4.18696	9.02053
7.35	54.0225	2.71109	8.57321	397.065	1.94430	4.18886	9.02462
7.36	54.1696	2.71293	8.57904	398.688	1.94518	4.19076	9.02871
7.37	54.3169	2.71477	8.58487	400.316	1.94606	4.19266	9.03280
7.38	54.4644	2.71662	8.59069	401.947	1.94694	4.19455	9.03689
7.39	54.6121	2.71846	8.59651	403.583	1.94782	4.19644	9.04097
7.40	54.7600	2.72029	8.60233	405.224	1.94870	4.19834	9.04504
7.41	54.9081	2.72213	8.60814	406.869	1.94957	4.20023	9.04911
7.42	55.0564	2.72397	8.61394	408.518	1.95045	4.20212	9.05318
7.43	55.2049	2.72580	8.61974	410.172	1.95132	4.20400	9.05725
7.44	55.3536	2.72764	8.62554	411.831	1.95220	4.20589	9.06131
7.45	55.5025	2.72947	8.63134	413.494	1.95307	4.20777	9.06537
7.46	55.6516	2.73130	8.63713	415.161	1.95395	4.20965	9.06942
7.47	55.8009	2.73313	8.64292	416.833	1.95482	4.21153	9.07347
7.48	55.9504	2.73496	8.64870	418.509	1.95569	4.21341	9.07752
7.49	56.1001	2.73679	8.65448	420.190	1.95656	4.21529	9.08156

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.50	56.2500	2.73861	8.66025	421.875	1.95743	4.21716	9.08560
7.51	56.4001	2.74044	8.66603	423.565	1.95830	4.21904	9.08964
7.52	56.5504	2.74226	8.67179	425.259	1.95917	4.22091	9.09367
7.53	56.7009	2.74408	8.67756	426.958	1.96004	4.22278	9.09770
7.54	56.8516	2.74591	8.68332	428.661	1.96091	4.22465	9.10173
7.55	57.0025	2.74773	8.68907	430.369	1.96177	4.22651	9.10575
7.56	57.1536	2.74955	8.69483	432.081	1.96264	4.22838	9.10977
7.57	57.3049	2.75136	8.70057	433.798	1.96350	4.23024	9.11378
7.58	57.4564	2.75318	8.70632	435.520	1.96437	4.23210	9.11779
7.59	57.6081	2.75500	8.71206	437.245	1.96523	4.23396	9.12180
7.60	57.7600	2.75681	8.71780	438.976	1.96610	4.23582	9.12581
7.61	57.9121	2.75862	8.72353	440.711	1.96696	4.23768	9.12981
7.62	58.0644	2.76043	8.72926	442.451	1.96782	4.23954	9.13380
7.63	58.2169	2.76225	8.73499	444.195	1.96868	4.24139	9.13780
7.64	58.3696	2.76405	8.74071	445.944	1.96954	4.24324	9.14179
7.65	58.5225	2.76586	8.74643	447.697	1.97040	4.24509	9.14577
7.66	58.6756	2.76767	8.75214	449.455	1.97126	4.24694	9.14976
7.67	58.8289	2.76948	8.75785	451.218	1.97211	4.24879	9.15374
7.68	58.9824	2.77128	8.76356	452.985	1.97297	4.25063	9.15771
7.69	59.1361	2.77308	8.76926	454.757	1.97383	4.25248	9.16169
7.70	59.2900	2.77489	8.77496	456.533	1.97468	4.25432	9.16566
7.71	59.4441	2.77669	8.78066	458.314	1.97554	4.25616	9.16962
7.72	59.5984	2.77849	8.78635	460.100	1.97639	4.25800	9.17359
7.73	59.7529	2.78029	8.79204	461.890	1.97724	4.25984	9.17754
7.74	59.9076	2.78209	8.79773	463.685	1.97809	4.26167	9.18150
7.75	60.0625	2.78388	8.80341	465.484	1.97895	4.26351	9.18545
7.76	60.2176	2.78568	8.80909	467.289	1.97980	4.26534	9.18940
7.77	60.3729	2.78747	8.81476	469.097	1.98065	4.26717	9.19335
7.78	60.5284	2.78927	8.82043	470.911	1.98150	4.26900	9.19729
7.79	60.6841	2.79106	8.82610	472.729	1.98234	4.27083	9.20123
7.80	60.8400	2.79285	8.83176	474.552	1.98319	4.27266	9.20516
7.81	60.9961	2.79464	8.83742	476.380	1.98404	4.27448	9.20910
7.82	61.1524	2.79643	8.84308	478.212	1.98489	4.27631	9.21302
7.83	61.3089	2.79821	8.84873	480.049	1.98573	4.27813	9.21695
7.84	61.4656	2.80000	8.85438	481.890	1.98658	4.27995	9.22087
7.85	61.6225	2.80179	8.86002	483.737	1.98742	4.28177	9.22479
7.86	61.7796	2.80357	8.86566	485.588	1.98826	4.28359	9.22871
7.87	61.9369	2.80535	8.87130	487.443	1.98911	4.28540	9.23262
7.88	62.0944	2.80713	8.87694	489.304	1.98995	4.28722	9.23653
7.89	62.2521	2.80891	8.88257	491.169	1.99079	4.28903	9.24043
7.90	62.4100	2.81069	8.88819	493.039	1.99163	4.29084	9.24434
7.91	62.5681	2.81247	8.89382	494.914	1.99247	4.29265	9.24823
7.92	62.7264	2.81425	8.89944	496.793	1.99331	4.29446	9.25213
7.93	62.8849	2.81603	8.90505	498.677	1.99415	4.29627	9.25602
7.94	63.0436	2.81780	8.91067	500.566	1.99499	4.29807	9.25991
7.95	63.2025	2.81957	8.91628	502.460	1.99582	4.29987	9.26380
7.96	63.3616	2.82135	8.92188	504.358	1.99666	4.30168	9.26768
7.97	63.5209	2.82312	8.92749	506.262	1.99750	4.30348	9.27156
7.98	63.6804	2.82489	8.93308	508.170	1.99833	4.30528	9.27544
7.99	63.8401	2.82666	8.93868	510.082	1.99917	4.30707	9.27931

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.00	64.0000	2.82843	8.94427	512.000	2.00000	4.30887	9.28318
8.01	64.1601	2.83019	8.94986	513.922	2.00083	4.31066	9.28704
8.02	64.3204	2.83196	8.95545	515.850	2.00167	4.31246	9.29091
8.03	64.4809	2.83373	8.96103	517.782	2.00250	4.31425	9.29477
8.04	64.6416	2.83549	8.96660	519.718	2.00333	4.31604	9.29862
8.05	64.8025	2.83725	8.97218	521.660	2.00416	4.31783	9.30248
8.06	64.9636	2.83901	8.97775	523.607	2.00499	4.31961	9.30633
8.07	65.1249	2.84077	8.98332	525.558	2.00582	4.32140	9.31018
8.08	65.2864	2.84253	8.98888	527.514	2.00664	4.32318	9.31402
8.09	65.4481	2.84429	8.99444	529.475	2.00747	4.32497	9.31786
8.10	65.6100	2.84605	9.00000	531.441	2.00830	4.32675	9.32170
8.11	65.7721	2.84781	9.00555	533.412	2.00912	4.32853	9.32553
8.12	65.9344	2.84956	9.01110	535.387	2.00995	4.33031	9.32936
8.13	66.0969	2.85132	9.01665	537.368	2.01078	4.33208	9.33319
8.14	66.2596	2.85307	9.02219	539.353	2.01160	4.33386	9.33702
8.15	66.4225	2.85482	9.02774	541.343	2.01242	4.33563	9.34084
8.16	66.5856	2.85657	9.03327	543.338	2.01325	4.33741	9.34466
8.17	66.7489	2.85832	9.03881	545.339	2.01407	4.33918	9.34847
8.18	66.9124	2.86007	9.04434	547.343	2.01489	4.34095	9.35229
8.19	67.0761	2.86182	9.04986	549.353	2.01571	4.34271	9.35610
8.20	67.2400	2.86356	9.05539	551.368	2.01653	4.34448	9.35990
8.21	67.4041	2.86531	9.06091	553.388	2.01735	4.34625	9.36370
8.22	67.5684	2.86705	9.06642	555.412	2.01817	4.34801	9.36751
8.23	67.7329	2.86880	9.07193	557.442	2.01899	4.34977	9.37130
8.24	67.8976	2.87054	9.07744	559.476	2.01980	4.35153	9.37510
8.25	68.0625	2.87228	9.08295	561.516	2.02062	4.35329	9.37889
8.26	68.2276	2.87402	9.08845	563.560	2.02144	4.35505	9.38268
8.27	68.3929	2.87576	9.09395	565.609	2.02225	4.35681	9.38646
8.28	68.5584	2.87750	9.09945	567.664	2.02307	4.35856	9.39024
8.29	68.7241	2.87924	9.10494	569.723	2.02388	4.36032	9.39402
8.30	68.8900	2.88097	9.11043	571.787	2.02469	4.36207	9.39780
8.31	69.0561	2.88271	9.11592	573.856	2.02551	4.36382	9.40157
8.32	69.2224	2.88444	9.12140	575.930	2.02632	4.36557	9.40534
8.33	69.3889	2.88617	9.12688	578.010	2.02713	4.36732	9.40911
8.34	69.5556	2.88791	9.13236	580.094	2.02794	4.36907	9.41287
8.35	69.7225	2.88964	9.13783	582.183	2.02875	4.37081	9.41663
8.36	69.8896	2.89137	9.14330	584.277	2.02956	4.37256	9.42039
8.37	70.0569	2.89310	9.14877	586.376	2.03037	4.37430	9.42414
8.38	70.2244	2.89482	9.15423	588.480	2.03118	4.37604	9.42789
8.39	70.3921	2.89655	9.15969	590.590	2.03199	4.37778	9.43164
8.40	70.5600	2.89828	9.16515	592.704	2.03279	4.37952	9.43539
8.41	70.7281	2.90000	9.17061	594.823	2.03360	4.38126	9.43913
8.42	70.8964	2.90172	9.17606	596.948	2.03440	4.38299	9.44287
8.43	71.0649	2.90345	9.18150	599.077	2.03521	4.38473	9.44661
8.44	71.2336	2.90517	9.18695	601.212	2.03601	4.38646	9.45034
8.45	71.4025	2.90689	9.19239	603.351	2.03682	4.38819	9.45407
8.46	71.5716	2.90861	9.19783	605.496	2.03762	4.38992	9.45780
8.47	71.7409	2.91033	9.20326	607.645	2.03842	4.39165	9.46152
8.48	71.9104	2.91204	9.20869	609.800	2.03923	4.39338	9.46525
8.49	72.0801	2.91376	9.21412	611.960	2.04003	4.39510	9.46897

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.50	72.2500	2.91548	9.21954	614.125	2.04083	4.39683	9.47268
8.51	72.4201	2.91719	9.22497	616.295	2.04163	4.39855	9.47640
8.52	72.5904	2.91890	9.23038	618.470	2.04243	4.40028	9.48011
8.53	72.7609	2.92062	9.23580	620.650	2.04323	4.40200	9.48381
8.54	72.9316	2.92233	9.24121	622.836	2.04402	4.40372	9.48752
8.55	73.1025	2.92404	9.24662	625.026	2.04482	4.40543	9.49122
8.56	73.2736	2.92575	9.25203	627.222	2.04562	4.40715	9.49492
8.57	73.4449	2.92746	9.25743	629.423	2.04641	4.40887	9.49861
8.58	73.6164	2.92916	9.26283	631.629	2.04721	4.41058	9.50231
8.59	73.7881	2.93087	9.26823	633.840	2.04801	4.41229	9.50600
8.60	73.9600	2.93258	9.27362	636.056	2.04880	4.41400	9.50969
8.61	74.1321	2.93428	9.27901	638.277	2.04959	4.41571	9.51337
8.62	74.3044	2.93598	9.28440	640.504	2.05039	4.41742	9.51705
8.63	74.4769	2.93769	9.28978	642.736	2.05118	4.41913	9.52073
8.64	74.6496	2.93939	9.29516	644.973	2.05197	4.42084	9.52441
8.65	74.8225	2.94109	9.30054	647.215	2.05276	4.42254	9.52808
8.66	74.9956	2.94279	9.30591	649.462	2.05355	4.42425	9.53175
8.67	75.1689	2.94449	9.31128	651.714	2.05434	4.42595	9.53542
8.68	75.3424	2.94618	9.31665	653.972	2.05513	4.42765	9.53908
8.69	75.5161	2.94788	9.32202	656.235	2.05592	4.42935	9.54274
8.70	75.6900	2.94958	9.32738	658.503	2.05671	4.43105	9.54640
8.71	75.8641	2.95127	9.33274	660.776	2.05750	4.43274	9.55006
8.72	76.0384	2.95296	9.33809	663.055	2.05828	4.43444	9.55371
8.73	76.2129	2.95466	9.34345	665.339	2.05907	4.43613	9.55736
8.74	76.3876	2.95635	9.34880	667.628	2.05986	4.43783	9.56101
8.75	76.5625	2.95804	9.35414	669.922	2.06064	4.43952	9.56466
8.76	76.7376	2.95973	9.35949	672.221	2.06143	4.44121	9.56830
8.77	76.9129	2.96142	9.36483	674.526	2.06221	4.44290	9.57194
8.78	77.0884	2.96311	9.37017	676.836	2.06299	4.44459	9.57557
8.79	77.2641	2.96479	9.37550	679.151	2.06378	4.44627	9.57921
8.80	77.4400	2.96648	9.38083	681.472	2.06456	4.44796	9.58284
8.81	77.6161	2.96816	9.38616	683.798	2.06534	4.44964	9.58647
8.82	77.7924	2.96985	9.39149	686.129	2.06612	4.45133	9.59009
8.83	77.9689	2.97153	9.39681	688.465	2.06690	4.45301	9.59372
8.84	78.1456	2.97321	9.40213	690.807	2.06768	4.45469	9.59734
8.85	78.3225	2.97489	9.40744	693.154	2.06846	4.45637	9.60095
8.86	78.4996	2.97658	9.41276	695.506	2.06924	4.45805	9.60457
8.87	78.6769	2.97825	9.41807	697.864	2.07002	4.45972	9.60818
8.88	78.8544	2.97993	9.42338	700.227	2.07080	4.46140	9.61179
8.89	79.0321	2.98161	9.42868	702.595	2.07157	4.46307	9.61540
8.90	79.2100	2.98329	9.43398	704.969	2.07235	4.46475	9.61900
8.91	79.3881	2.98496	9.43928	707.348	2.07313	4.46642	9.62260
8.92	79.5664	2.98664	9.44458	709.732	2.07390	4.46809	9.62620
8.93	79.7449	2.98831	9.44987	712.122	2.07468	4.46976	9.62980
8.94	79.9236	2.98998	9.45516	714.517	2.07545	4.47142	9.63339
8.95	80.1025	2.99166	9.46044	716.917	2.07622	4.47309	9.63698
8.96	80.2816	2.99333	9.46573	719.323	2.07700	4.47476	9.64057
8.97	80.4609	2.99500	9.47101	721.734	2.07777	4.47642	9.64415
8.98	80.6404	2.99666	9.47629	724.151	2.07854	4.47808	9.64774
8.99	80.8201	2.99833	9.48156	726.573	2.07931	4.47974	9.65132

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.00	81.0000	3.00000	9.48683	729.000	2.08008	4.48140	9.65489
9.01	81.1801	3.00167	9.49210	731.433	2.08085	4.48306	9.65847
9.02	81.3604	3.00333	9.49737	733.871	2.08162	4.48472	9.66204
9.03	81.5409	3.00500	9.50263	736.314	2.08239	4.48638	9.66561
9.04	81.7216	3.00666	9.50789	738.763	2.08316	4.48803	9.66918
9.05	81.9025	3.00832	9.51315	741.218	2.08393	4.48969	9.67274
9.06	82.0836	3.00998	9.51840	743.677	2.08470	4.49134	9.67630
9.07	82.2649	3.01164	9.52365	746.143	2.08546	4.49299	9.67986
9.08	82.4464	3.01330	9.52890	748.613	2.08623	4.49464	9.68342
9.09	82.6281	3.01496	9.53415	751.089	2.08699	4.49629	9.68697
9.10	82.8100	3.01662	9.53939	753.571	2.08776	4.49794	9.69052
9.11	82.9921	3.01828	9.54463	756.058	2.08852	4.49959	9.69407
9.12	83.1744	3.01993	9.54987	758.551	2.08929	4.50123	9.69762
9.13	83.3569	3.02159	9.55510	761.048	2.09005	4.50288	9.70116
9.14	83.5396	3.02324	9.56033	763.552	2.09081	4.50452	9.70470
9.15	83.7225	3.02490	9.56556	766.061	2.09158	4.50616	9.70824
9.16	83.9056	3.02655	9.57079	768.575	2.09234	4.50781	9.71177
9.17	84.0889	3.02820	9.57601	771.095	2.09310	4.50945	9.71531
9.18	84.2724	3.02985	9.58123	773.621	2.09386	4.51108	9.71884
9.19	84.4561	3.03150	9.58645	776.152	2.09462	4.51272	9.72236
9.20	84.6400	3.03315	9.59166	778.688	2.09538	4.51436	9.72589
9.21	84.8241	3.03480	9.59687	781.230	2.09614	4.51599	9.72941
9.22	85.0084	3.03645	9.60208	783.777	2.09690	4.51763	9.73293
9.23	85.1929	3.03809	9.60729	786.330	2.09765	4.51926	9.73645
9.24	85.3776	3.03974	9.61249	788.889	2.09841	4.52089	9.73996
9.25	85.5625	3.04138	9.61769	791.453	2.09917	4.52252	9.74348
9.26	85.7476	3.04302	9.62289	794.023	2.09992	4.52415	9.74699
9.27	85.9329	3.04467	9.62808	796.598	2.10068	4.52578	9.75049
9.28	86.1184	3.04631	9.63328	799.179	2.10144	4.52740	9.75400
9.29	86.3041	3.04795	9.63846	801.765	2.10219	4.52903	9.75750
9.30	86.4900	3.04959	9.64365	804.357	2.10294	4.53065	9.76100
9.31	86.6761	3.05123	9.64883	806.954	2.10370	4.53228	9.76450
9.32	86.8624	3.05287	9.65401	809.558	2.10445	4.53390	9.76799
9.33	87.0489	3.05450	9.65919	812.166	2.10520	4.53552	9.77148
9.34	87.2356	3.05614	9.66437	814.781	2.10595	4.53714	9.77497
9.35	87.4225	3.05778	9.66954	817.400	2.10671	4.53876	9.77846
9.36	87.6096	3.05941	9.67471	820.026	2.10746	4.54038	9.78195
9.37	87.7969	3.06105	9.67988	822.657	2.10821	4.54199	9.78543
9.38	87.9844	3.06268	9.68504	825.294	2.10896	4.54361	9.78891
9.39	88.1721	3.06431	9.69020	827.936	2.10971	4.54522	9.79239
9.40	88.3600	3.06594	9.69536	830.584	2.11045	4.54684	9.79586
9.41	88.5481	3.06757	9.70052	833.238	2.11120	4.54845	9.79933
9.42	88.7364	3.06920	9.70567	835.897	2.11195	4.55006	9.80280
9.43	88.9249	3.07083	9.71082	838.562	2.11270	4.55167	9.80627
9.44	89.1136	3.07246	9.71597	841.232	2.11344	4.55328	9.80974
9.45	89.3025	3.07409	9.72111	843.909	2.11419	4.55488	9.81320
9.46	89.4916	3.07571	9.72625	846.591	2.11494	4.55649	9.81666
9.47	89.6809	3.07734	9.73139	849.278	2.11568	4.55809	9.82012
9.48	89.8704	3.07896	9.73653	851.971	2.11642	4.55970	9.82357
9.49	90.0601	3.08058	9.74166	854.670	2.11717	4.56130	9.82703

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.50	90.2500	3.08221	9.74679	857.375	2.11791	4.56290	9.83048
9.51	90.4401	3.08383	9.75192	860.085	2.11865	4.56450	9.83392
9.52	90.6304	3.08545	9.75705	862.801	2.11940	4.56610	9.83737
9.53	90.8209	3.08707	9.76217	865.523	2.12014	4.56770	9.84081
9.54	91.0116	3.08869	9.76729	868.251	2.12088	4.56930	9.84425
9.55	91.2025	3.09031	9.77241	870.984	2.12162	4.57089	9.84769
9.56	91.3936	3.09192	9.77753	873.723	2.12236	4.57249	9.85113
9.57	91.5849	3.09354	9.78264	876.467	2.12310	4.57408	9.85456
9.58	91.7764	3.09516	9.78775	879.218	2.12384	4.57567	9.85799
9.59	91.9681	3.09677	9.79285	881.974	2.12458	4.57727	9.86142
9.60	92.1600	3.09839	9.79796	884.736	2.12532	4.57886	9.86485
9.61	92.3521	3.10000	9.80306	887.504	2.12605	4.58045	9.86827
9.62	92.5444	3.10161	9.80816	890.277	2.12679	4.58204	9.87169
9.63	92.7369	3.10322	9.81326	893.056	2.12753	4.58362	9.87511
9.64	92.9296	3.10483	9.81835	895.841	2.12826	4.58521	9.87853
9.65	93.1225	3.10644	9.82344	898.632	2.12900	4.58679	9.88195
9.66	93.3156	3.10805	9.82853	901.429	2.12974	4.58838	9.88536
9.67	93.5089	3.10966	9.83362	904.231	2.13047	4.58996	9.88877
9.68	93.7024	3.11127	9.83870	907.039	2.13120	4.59154	9.89217
9.69	93.8961	3.11288	9.84378	909.853	2.13194	4.59312	9.89558
9.70	94.0900	3.11448	9.84886	912.673	2.13267	4.59470	9.89898
9.71	94.2841	3.11609	9.85393	915.499	2.13340	4.59628	9.90238
9.72	94.4784	3.11769	9.85901	918.330	2.13414	4.59786	9.90578
9.73	94.6729	3.11929	9.86408	921.167	2.13487	4.59943	9.90918
9.74	94.8676	3.12090	9.86914	924.010	2.13560	4.60101	9.91257
9.75	95.0625	3.12250	9.87421	926.859	2.13633	4.60258	9.91596
9.76	95.2576	3.12410	9.87927	929.714	2.13706	4.60416	9.91935
9.77	95.4529	3.12570	9.88433	932.575	2.13779	4.60573	9.92274
9.78	95.6484	3.12730	9.88939	935.441	2.13852	4.60730	9.92612
9.79	95.8441	3.12890	9.89444	938.314	2.13925	4.60887	9.92950
9.80	96.0400	3.13050	9.89949	941.192	2.13997	4.61044	9.93288
9.81	96.2361	3.13209	9.90454	944.076	2.14070	4.61200	9.93626
9.82	96.4324	3.13369	9.90959	946.966	2.14143	4.61357	9.93964
9.83	96.6289	3.13528	9.91464	949.862	2.14216	4.61514	9.94301
9.84	96.8256	3.13688	9.91968	952.764	2.14288	4.61670	9.94638
9.85	97.0225	3.13847	9.92472	955.672	2.14361	4.61826	9.94975
9.86	97.2196	3.14006	9.92975	958.585	2.14433	4.61983	9.95311
9.87	97.4169	3.14166	9.93479	961.505	2.14506	4.62139	9.95648
9.88	97.6144	3.14325	9.93982	964.430	2.14578	4.62295	9.95984
9.89	97.8121	3.14484	9.94485	967.362	2.14651	4.62451	9.96320
9.90	98.0100	3.14643	9.94987	970.299	2.14723	4.62607	9.96655
9.91	98.2081	3.14802	9.95490	973.242	2.14795	4.62762	9.96991
9.92	98.4064	3.14960	9.95992	976.191	2.14867	4.62918	9.97326
9.93	98.6049	3.15119	9.96494	979.147	2.14940	4.63073	9.97661
9.94	98.8036	3.15278	9.96995	982.108	2.15012	4.63229	9.97996
9.95	99.0025	3.15436	9.97497	985.075	2.15084	4.63384	9.98331
9.96	99.2016	3.15595	9.97998	988.048	2.15156	4.63539	9.98665
9.97	99.4009	3.15753	9.98499	991.027	2.15228	4.63694	9.98999
9.98	99.6004	3.15911	9.98999	994.012	2.15300	4.63849	9.99333
9.99	99.8001	3.16070	9.99500	997.003	2.15372	4.64004	9.99667

TABLE II — IMPORTANT NUMBERS

A. *Units of Length*

English Units		Metric Units	
12 inches (in.)	= **1 foot** (ft.)	10 millimeters (mm.)	= **1 centimeter (cm.)**
3 feet	= 1 yard (yd.)	10 centimeters	= 1 decimeter (dm.)
$5\frac{1}{2}$ yards	= 1 rod (rd.)	10 decimeters	= **1 meter (m.)**
320 rods	= **1 mile** (mi.)	10 meters	= 1 dekameter (Dm.)
		1000 meters	= **1 kilometer (Km.)**

English to Metric	Metric to English
1 in. = 2.5400 cm.	**1 cm. = 0.3937 in.**
1 ft. = 30.480 cm.	1 m. = 39.37 in. = **3.2808 ft.**
1 mi. = 1.6093 Km.	1 Km. = 0.6214 mi.

B. *Units of Area or Surface*

1 square yard = 9 square feet = 1296 square inches
1 acre (A.) = 160 square rods = 4840 square yards
1 square mile = 640 acres = 102400 square rods

C. *Units of Measurement of Capacity*

Dry Measure		Liquid Measure	
2 pints (pt.)	= 1 quart (qt.)	4 gills (gi.)	= 1 pint (pt.)
8 quarts	= 1 peck (pk.)	2 pints	= 1 quart (qt.)
4 pecks	= 1 bushel (bu.)	4 quarts	= 1 gallon (gal.)
		1 gallon	= 231 cu. in.

D. *Metric Units to English Units*

1 liter = 1000 cu. cm. = 61.02 cu. in. = 1.0567 liquid quarts
1 quart = .94636 liter = 946.36 cu. cm.
1000 grams = 1 kilogram (Kg.) = 2.2046 pounds (lb.)
1 pound = .453593 kilogram = 453.59 grams

E. *Other Numbers*

π = ratio of circumference to diameter of a circle
= 3.14159265
1 radian = angle subtended by an arc equal to the radius
= $57° 17' 44''.8 = 57°.2957795 = 180°/\pi$
1 degree = 0.01745329 radian, or $\pi/180$ radians
Weight of 1 cu. ft. of water = 62.425 lb.

INDEX

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Macmillan
Logarithmic and Trigonometric Tables

Prepared under the direction of E. R. HEDRICK, Head of the Department of Mathematics in the University of Missouri.

This volume contains the usual five-place tables of logarithms, trigonometric functions and logarithms of trigonometric functions. It also contains four-place tables for use in solving those problems for which extreme accuracy is not demanded. By the use of a complete and handily arranged table (not usually found in such volumes) of squares, cubes, square roots, cube roots and reciprocals, the Pythagorean theorem and the Cosine Law become practicable for actual computation. The inclusion of tables of the trigonometric functions in radian measure, and of tables for conversion of degrees to radians and conversely, is another feature not always found in similar volumes

The style of type, form of rulings, and the arrangement have been selected in order to make possible the greatest rapidity of reading with the least strain upon the eyes. These selections have been guided by a thorough study of the results of recent psychological experiments along these lines.

These tables are bound with the text of Kenyon and Ingold's Trigonometry and are also published separately at sixty cents.
